MATH4/68181: EXTREME VALUES AND FINANCIAL RISK
SEMESTER 1
SOLUTION TO QUIZ PROBLEM 4

Suppose a company has N systems functioning independently at a given time, where N is a geometric random variable with the probability mass function

$$\Pr(N = n) = (1 - p)p^{n-1}$$

for $n = 1, 2, \ldots$. Suppose too that each system is made of α parallel units, so the system will fail if all of the units fail. Assume that the failure times of the units for the ith system, say $Z_{i,1}, Z_{i,2}, \ldots, Z_{i,\alpha}$, are independent and identical exponential random variables with probability density function $\beta \exp(-\beta z)$, $z > 0$. Let Y_i denote the failure time of ith system. Let X denote the time to failure of the first out of the N functioning systems.

We can write $X = \min(Y_1, Y_2, \ldots, Y_N)$. Then the cumulative distribution function of X, say $G(x)$, can be derived as: the conditional cumulative distribution function of X given N is

$$\begin{aligned}
G(x|N) &= 1 - \Pr(X > x|N) \\
&= 1 - \Pr(Y_1 > x, Y_2 > x, \ldots, Y_N > x) \\
&= 1 - \Pr^N(Y_1 > x) \\
&= 1 - [1 - \Pr(Y_1 \leq x)]^N \\
&= 1 - [1 - \Pr(Z_{1,1} \leq x, Z_{1,2} \leq x, \ldots, Z_{1,\alpha} \leq x)]^N \\
&= 1 - [1 - \Pr^\alpha(Z_{1,1} \leq x)]^N \\
&= 1 - [1 - \{1 - \exp(-\beta x)\}^\alpha]^N,
\end{aligned}$$

and so the unconditional cumulative distribution function of X is

$$\begin{aligned}
G(x) &= (1-p)\sum_{n=1}^{\infty} \{1 - [1 - \{1 - \exp(-\beta x)\}^\alpha]^n\} p^{n-1} \\
&= (1-p)\sum_{n=1}^{\infty} p^{n-1} - (1-p)\sum_{n=1}^{\infty} [1 - \{1 - \exp(-\beta x)\}^\alpha]^n p^{n-1} \\
&= 1 - (1-p)[1 - \{1 - \exp(-\beta x)\}^\alpha]\sum_{n=1}^{\infty} [1 - \{1 - \exp(-\beta x)\}^\alpha]^{n-1} p^{n-1} \\
&= 1 - (1-p)[1 - \{1 - \exp(-\beta x)\}^\alpha]\sum_{m=0}^{\infty} \{p[1 - \{1 - \exp(-\beta x)\}^\alpha]\}^m \\
&= 1 - (1-p)[1 - \{1 - \exp(-\beta x)\}^\alpha]\frac{1}{1 - p[1 - \{1 - \exp(-\beta x)\}^\alpha]} \\
&= \frac{\{1 - \exp(-\beta x)\}^\alpha}{1 - p + p\{1 - \exp(-\beta x)\}^\alpha}
\end{aligned}$$

for $x > 0$, $0 < p < 1$, $\beta > 0$ and $\alpha > 0$.

Setting $G(x) = q$ and solving for x in terms of q, we have

$$x = -\frac{1}{\beta} \log \left\{ 1 - \left[\frac{q(1-p)}{1-pq} \right]^{1/\alpha} \right\}.$$

Hence,

$$\mathrm{VaR}_q(X) = -\frac{1}{\beta} \log \left\{ 1 - \left[\frac{q(1-p)}{1-pq} \right]^{1/\alpha} \right\}.$$

for $0 < q < 1$.