MATH4/68181: EXTREME VALUES AND FINANCIAL RISK
SEMESTER 1
SOLUTIONS TO QUIZ PROBLEM 3

Consider a class of distributions defined by the cdf

$$F(x) = [G(x)]^a \left\{ 1 + \lambda - \lambda [G(x)]^b \right\}$$

where $a > 0$, $b > 0$, $-\infty < \lambda < \infty$ and $G(\cdot)$ is a valid cdf. Assume that F and G have the same upper end points.

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function say $h(t)$ such that

$$\lim_{t \to w(G)} \frac{1 - G(t + xh(t))}{1 - G(t)} = e^{-x}$$

for every $x > 0$. But

$$\begin{aligned}
& \lim_{t \to w(F)} \frac{1 - F(t + xh(t))}{1 - F(t)} \\
= & \lim_{t \to w(F)} \frac{1 - [G(t + xh(t))]^a \left\{ 1 + \lambda - \lambda [G(t + xh(t))]^b \right\}}{1 - [G(t)]^a \left\{ 1 + \lambda - \lambda [G(t)]^b \right\}} \\
= & \lim_{t \to w(F)} \frac{1 - [G(t + xh(t))]^a - \lambda [G(t + xh(t))]^a \left\{ 1 - [G(t + xh(t))]^b \right\}}{1 - [G(t)]^a - \lambda [G(t)]^a \left\{ 1 - [G(t)]^b \right\}} \\
= & \lim_{t \to w(G)} \frac{1 - [G(t + xh(t))]^a - \lambda [G(t + xh(t))]^a \left\{ 1 - [G(t + xh(t))]^b \right\}}{1 - [G(t)]^a - \lambda [G(t)]^a \left\{ 1 - [G(t)]^b \right\}} \\
= & \lim_{t \to w(G)} \frac{1 - 1^a - \lambda \cdot 1^a \cdot \left\{ 1 - [G(t + xh(t))]^b \right\}}{1 - 1^a - \lambda \cdot 1^a \cdot \left\{ 1 - [G(t)]^b \right\}} \\
= & \lim_{t \to w(G)} \frac{1 - [G(t + xh(t))]^b}{1 - [G(t)]^b} \\
= & \lim_{t \to w(G)} \frac{1 - \{1 - [1 - G(t + xh(t))]\}^b}{1 - \{1 - [1 - G(t)]\}^b} \\
= & \lim_{t \to w(G)} \frac{1 - \{1 - b[1 - G(t + xh(t))]\}}{1 - \{1 - b[1 - G(t)]\}} \\
= & \lim_{t \to w(G)} \frac{1 - G(t + xh(t))}{1 - G(t)} \\
= & \exp(-x)
\end{aligned}$$

for every $x > 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left[-\exp(-x)\right]$$

for some suitable norming constants $a_n > 0$ and b_n.

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t\to\infty} \frac{1 - G\left(tx\right)}{1 - G(t)} = x^{-\beta}$$

for every $x > 0$. But

$$\begin{aligned}
& \lim_{t\to\infty} \frac{1 - F\left(tx\right)}{1 - F(t)} \\
= & \lim_{t\to\infty} \frac{1 - \left[G\left(tx\right)\right]^a \left\{1 + \lambda - \lambda \left[G\left(tx\right)\right]^b\right\}}{1 - \left[G(t)\right]^a \left\{1 + \lambda - \lambda \left[G(t)\right]^b\right\}} \\
= & \lim_{t\to\infty} \frac{1 - \left[G\left(tx\right)\right]^a - \lambda \left[G\left(tx\right)\right]^a \left\{1 - \left[G\left(tx\right)\right]^b\right\}}{1 - \left[G\left(t\right)\right]^a - \lambda \left[G\left(t\right)\right]^a \left\{1 - \left[G\left(t\right)\right]^b\right\}} \\
= & \lim_{t\to\infty} \frac{1 - 1^a - \lambda \cdot 1^a \cdot \left\{1 - \left[G\left(tx\right)\right]^b\right\}}{1 - 1^a - \lambda \cdot 1^a \cdot \left\{1 - \left[G\left(t\right)\right]^b\right\}} \\
= & \lim_{t\to\infty} \frac{1 - \left[G\left(tx\right)\right]^b}{1 - \left[G\left(t\right)\right]^b} \\
= & \lim_{t\to\infty} \frac{1 - \left\{1 - \left[1 - G\left(tx\right)\right]\right\}^b}{1 - \left\{1 - \left[1 - G\left(t\right)\right]\right\}^b} \\
= & \lim_{t\to\infty} \frac{1 - \left\{1 - b\left[1 - G\left(tx\right)\right]\right\}}{1 - \left\{1 - b\left[1 - G\left(t\right)\right]\right\}} \\
= & \lim_{t\to\infty} \frac{1 - G\left(tx\right)}{1 - G\left(t\right)} \\
= & x^{-\beta}.
\end{aligned}$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left(-x^{-\beta}\right)$$

for some suitable norming constants $a_n > 0$ and b_n.

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t\to 0} \frac{1 - G\left(w(G) - tx\right)}{1 - G\left(w(G) - t\right)} = x^{\beta}$$

for every $x > 0$. But

$$
\begin{aligned}
& \lim_{t \to 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} \\
= & \lim_{t \to 0} \frac{1 - \left[G\left(w(F) - tx\right)\right]^a \left\{1 + \lambda - \lambda \left[G\left(w(F) - tx\right)\right]^b\right\}}{1 - \left[G\left(w(F) - t\right)\right]^a \left\{1 + \lambda - \lambda \left[G\left(w(F) - t\right)\right]^b\right\}} \\
= & \lim_{t \to 0} \frac{1 - \left[G\left(w(F) - tx\right)\right]^a - \lambda \left[G\left(w(F) - tx\right)\right]^a \left\{1 - \left[G\left(w(F) - tx\right)\right]^b\right\}}{1 - \left[G\left(w(F) - t\right)\right]^a - \lambda \left[G\left(w(F) - t\right)\right]^a \left\{1 - \left[G\left(w(F) - t\right)\right]^b\right\}} \\
= & \lim_{t \to 0} \frac{1 - \left[G\left(w(G) - tx\right)\right]^a - \lambda \left[G\left(w(G) - tx\right)\right]^a \left\{1 - \left[G\left(w(G) - tx\right)\right]^b\right\}}{1 - \left[G\left(w(G) - t\right)\right]^a - \lambda \left[G\left(w(G) - t\right)\right]^a \left\{1 - \left[G\left(w(G) - t\right)\right]^b\right\}} \\
= & \lim_{t \to 0} \frac{1 - 1^a - \lambda \cdot 1^a \cdot \left\{1 - \left[G\left(w(G) - tx\right)\right]^b\right\}}{1 - 1^a - \lambda \cdot 1^a \cdot \left\{1 - \left[G\left(w(G) - t\right)\right]^b\right\}} \\
= & \lim_{t \to 0} \frac{1 - \left[G\left(w(G) - tx\right)\right]^b}{1 - \left[G\left(w(G) - t\right)\right]^b} \\
= & \lim_{t \to 0} \frac{1 - \left\{1 - \left[1 - G\left(w(G) - tx\right)\right]\right\}^b}{1 - \left\{1 - \left[1 - G\left(w(G) - t\right)\right]\right\}^b} \\
= & \lim_{t \to 0} \frac{1 - \left\{1 - b\left[1 - G\left(w(G) - tx\right)\right]\right\}}{1 - \left\{1 - b\left[1 - G\left(w(G) - t\right)\right]\right\}} \\
= & \lim_{t \to 0} \frac{1 - G\left(w(G) - tx\right)}{1 - G\left(w(G) - t\right)} \\
= & x^\beta
\end{aligned}
$$

for every $x > 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$
\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \le x\right) = \exp\left(-(-x)^\beta\right)
$$

for some suitable norming constants $a_n > 0$ and b_n.