MATH4/68181: EXTREME VALUES AND FINANCIAL RISK
SEMESTER 1
SOLUTIONS TO QUIZ PROBLEM 1

Suppose X is a random variable with cumulative distribution function

$$F(x) = 1 - \left[1 + \left(\frac{x}{\lambda} \right)^c \right]^{-k}$$

for $x > 0$, $c > 0$, $\lambda > 0$ and $k > 0$. Setting $F(x) = 1$ implies that

$$\left[1 + \left(\frac{x}{\lambda} \right)^c \right]^{-k} = 0$$

which implies that

$$1 + \left(\frac{x}{\lambda} \right)^c = \infty$$

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which implies that $x = \infty$. So, $w(F) = \infty$.

First check condition I. We have

$$\begin{aligned}
\lim_{t \to \infty} \frac{1 - F\left(t + x\gamma(t)\right)}{1 - F(t)} &= \lim_{t \to \infty} \frac{\left[1 + \left(\frac{t + x\gamma(t)}{\lambda} \right)^c \right]^{-k}}{\left[1 + \left(\frac{t}{\lambda} \right)^c \right]^{-k}} \\
&= \lim_{t \to \infty} \left[\frac{1 + \left(\frac{t + x\gamma(t)}{\lambda} \right)^c}{1 + \left(\frac{t}{\lambda} \right)^c} \right]^{-k} \\
&= \lim_{t \to \infty} \left[\frac{t^{-c} + \left(1 + \frac{x\gamma(t)}{t\lambda} \right)^c}{t^{-c} + \left(\frac{1}{\lambda} \right)^c} \right]^{-k} \\
&= \lim_{t \to \infty} \left[\frac{0 + \left(\frac{x\gamma(t)}{t\lambda} \right)^c}{0 + \left(\frac{1}{\lambda} \right)^c} \right]^{-k}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{t\to\infty} \left[\frac{\left(\frac{x\gamma(t)}{t\lambda}\right)^c}{\left(\frac{1}{\lambda}\right)^c} \right]^{-k} \\
&= \lim_{t\to\infty} \left(\frac{x\gamma(t)}{t} \right)^{-ck}.
\end{aligned}
$$

This can be equal to $\exp(-x)$ and hence F cannot belong to the Gumbel domain of attraction.

Now check condition II. We have

$$
\begin{aligned}
\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} &= \lim_{t\to\infty} \frac{\left[1+\left(\frac{tx}{\lambda}\right)^c\right]^{-k}}{\left[1+\left(\frac{t}{\lambda}\right)^c\right]^{-k}} \\
&= \lim_{t\to\infty} \left[\frac{1+\left(\frac{tx}{\lambda}\right)^c}{1+\left(\frac{t}{\lambda}\right)^c} \right]^{-k} \\
&= \lim_{t\to\infty} \left[\frac{t^{-c}+\left(\frac{x}{\lambda}\right)^c}{t^{-c}+\left(\frac{1}{\lambda}\right)^c} \right]^{-k} \\
&= \left[\frac{0+\left(\frac{x}{\lambda}\right)^c}{0+\left(\frac{1}{\lambda}\right)^c} \right]^{-k} \\
&= \left[\frac{\left(\frac{x}{\lambda}\right)^c}{\left(\frac{1}{\lambda}\right)^c} \right]^{-k} \\
&= x^{-ck}.
\end{aligned}
$$

Hence, F belongs to Fréchet domain of attraction.