SOLUTIONS TO MATH68181 EXTREME VALUES AND FINANCIAL RISK EXAM

Solutions to Question A1

ILO: bivariate extreme value distributions

a) The marginal cumulative distribution functions of X and Y are

$$F_X(x) = F_{X,Y}(x, \infty) = [H_{U,V}(x, \infty)]^\alpha = [H_U(x)]^\alpha$$

and

$$F_Y(y) = F_{X,Y}(\infty, y) = [H_{U,V}(\infty, y)]^\alpha = [H_V(y)]^\alpha ,$$

respectively.

(2 marks)

UNSEEN

b) Setting $F_X(x) = 1$ gives

$$F_X(x) = [H_U(x)]^\alpha = 1$$

which implies

$$H_U(x) = 1$$

which implies $x = w(H_U)$. Hence, $w(F_X) = w(H_U)$.

(1 marks)

UNSEEN

c) Setting $F_Y(y) = 1$ gives

$$F_Y(y) = [H_V(y)]^\alpha = 1$$

which implies

$$H_V(y) = 1$$

which implies $y = w(H_V)$. Hence, $w(F_Y) = w(H_V)$.

(1 marks)

UNSEEN

d) Suppose that $H_U(u) = H_{U,V}(u, \infty)$ belongs to the Gumbel max domain of attraction. Then there exists $\gamma(t) > 0$ such that

$$\lim_{t \to w(H)} \frac{1 - H_U(t + \gamma_U(t)u)}{1 - H_U(t)} = \exp(-u).$$

So,

$$\begin{aligned}
\lim_{t \to w(F_X)} \frac{1 - F_X\left(t + \gamma_U(t)x\right)}{1 - F_X(t)} &= \lim_{t \to w(F_X)} \frac{1 - \left[H_U\left(t + \gamma_U(t)x\right)\right]^{\alpha}}{1 - \left[H_V(t)\right]^{\alpha}} \\
&= \lim_{t \to w(H_U)} \frac{1 - \left[H_U\left(t + \gamma_U(t)x\right)\right]^{\alpha}}{1 - \left[H_V(t)\right]^{\alpha}} \\
&= \lim_{t \to w(H_U)} \frac{1 - \left\{1 - \left[1 - H_U\left(t + \gamma_U(t)x\right)\right]\right\}^{\alpha}}{1 - \left\{1 - \left[1 - H_U\left(t\right)\right]\right\}^{\alpha}} \\
&= \lim_{t \to w(H_U)} \frac{1 - \left\{1 - \alpha\left[1 - H_U\left(t + \gamma_U(t)x\right)\right]\right\}}{1 - \left\{1 - \alpha\left[1 - H_U\left(t\right)\right]\right\}} \\
&= \lim_{t \to w(H_U)} \frac{1 - H_U\left(t + \gamma_U(t)x\right)}{1 - H_U\left(t\right)} \\
&= \exp(-x)
\end{aligned}$$

and $F_X(x)$ also belongs to the Gumbel max domain of attraction.

(3 marks)

UNSEEN

e) Suppose that $H_V(v) = H_{U,V}(\infty, v)$ belongs to the Gumbel max domain of attraction. Then there exists $\gamma_V(t) > 0$ such that

$$\lim_{t \to w(H)} \frac{1 - H_V\left(t + \gamma_V(t)v\right)}{1 - H_V(t)} = \exp(-v).$$

So,

$$\begin{aligned}
\lim_{t \to w(F_Y)} \frac{1 - F_Y\left(t + \gamma_V(t)y\right)}{1 - F_Y(t)} &= \lim_{t \to w(F_Y)} \frac{1 - \left[H_V\left(t + \gamma_V(t)y\right)\right]^{\alpha}}{1 - \left[H_V(t)\right]^{\alpha}} \\
&= \lim_{t \to w(H_V)} \frac{1 - \left[H_V\left(t + \gamma_V(t)y\right)\right]^{\alpha}}{1 - \left[H_V(t)\right]^{\alpha}} \\
&= \lim_{t \to w(H_V)} \frac{1 - \left\{1 - \left[1 - H_V\left(t + \gamma_V(t)y\right)\right]\right\}^{\alpha}}{1 - \left\{1 - \left[1 - H_V\left(t\right)\right]\right\}^{\alpha}} \\
&= \lim_{t \to w(H_V)} \frac{1 - \left\{1 - \alpha\left[1 - H_V\left(t + \gamma_V(t)y\right)\right]\right\}}{1 - \left\{1 - \alpha\left[1 - H_V\left(t\right)\right]\right\}} \\
&= \lim_{t \to w(H_V)} \frac{1 - H_V\left(t + \gamma_V(t)y\right)}{1 - H_V\left(t\right)} \\
&= \exp(-y)
\end{aligned}$$

and $F_Y(y)$ also belongs to the Gumbel max domain of attraction.

(3 marks)

UNSEEN

f) If a_n and b_n satisfy

$$\left[H_U\left(a_n u + b_n\right)\right]^n \to \exp(u)$$

as $n \to \infty$ then

$$a_n = \gamma_U\left(F_U^{-1}\left(1 - \frac{1}{n}\right)\right)$$

and

$$b_n = F_U^{-1}\left(1 - \frac{1}{n}\right).$$

F_X also belongs to the Gumbel max domain of attraction with

$$a_n^{'} = \gamma_U\left(F_X^{-1}\left(1 - \frac{1}{n}\right)\right)$$

and

$$b_n^{'} = F_X^{-1}\left(1 - \frac{1}{n}\right).$$

Setting

$$F_X(x) = 1 - \frac{1}{n}$$

implies

$$\left[H_U(x)\right]^\alpha = 1 - \frac{1}{n}$$

which implies

$$x = H_U^{-1}\left(\left(1 - \frac{1}{n}\right)^{1/\alpha}\right)$$

which behaves as

$$x = H_U^{-1}\left(1 - \frac{1}{\alpha n}\right).$$

Hence, $a_n^{'} = a_{\alpha n}$ and $b_n^{'} = b_{\alpha n}$.

(2 marks)

UNSEEN

g) If c_n and d_n satisfy

$$[H_V(c_n v + d_n)]^n \to \exp(v)$$

as $n \to \infty$ then

$$c_n = \gamma_V \left(F_V^{-1} \left(1 - \frac{1}{n} \right) \right)$$

and

$$d_n = F_V^{-1} \left(1 - \frac{1}{n} \right).$$

F_Y also belongs to the Gumbel max domain of attraction with

$$c_n^{'} = \gamma_V \left(F_Y^{-1} \left(1 - \frac{1}{n} \right) \right)$$

and

$$d_n^{'} = F_Y^{-1} \left(1 - \frac{1}{n} \right).$$

Setting

$$F_Y(y) = 1 - \frac{1}{n}$$

implies

$$[H_V(y)]^\alpha = 1 - \frac{1}{n}$$

which implies

$$y = H_V^{-1} \left(\left(1 - \frac{1}{n} \right)^{1/\alpha} \right)$$

which behaves as

$$y = H_V^{-1} \left(1 - \frac{1}{\alpha n} \right).$$

Hence, $c_n^{'} = c_{\alpha n}$ and $d_n^{'} = d_{\alpha n}$.

(2 marks)

UNSEEN

h) The limiting cumulative distribution function is

$$\begin{aligned}
& \lim_{n\to\infty} \left[F_{X,Y}\left(a_{\alpha n}x + b_{\alpha n}, c_{\alpha n}y + d_{\alpha n}\right)\right]^{n} \\
= & \lim_{n\to\infty} \left[H_{X,Y}\left(a_{\alpha n}x + b_{\alpha n}, c_{\alpha n}y + d_{\alpha n}\right)\right]^{\alpha n} \\
= & \lim_{n\to\infty} \left[H_{X,Y}\left(a_{m}x + b_{m}, c_{m}y + d_{m}\right)\right]^{m} \qquad \text{set } m = \alpha n \\
= & \ G(x, y).
\end{aligned}$$

(5 marks)

UNSEEN

h) This is trivial since the limiting distributions for (X, Y) and (U, V) are the same.

(1 marks)

UNSEEN

Solutions to Question A2

ILO: checking a function is a copula

$C(u_1, u_2)$ is a valid copula if

$$C(u, 0) = 0,$$

$$C(0, u) = 0,$$

$$C(1, u) = u,$$

$$C(u, 1) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) \geq 0.$$

(4 marks)

SEEN

a) for C defined by

$$C(u_1, u_2) = \min\left[C_1(u_1, u_2), C_2(u_1, u_2)\right],$$

we have

$$C(u, 0) = \min\left[C_1(u, 0), C_2(u, 0)\right] = \min(0, 0) = 0,$$

$$C(0, u) = \min\left[C_1(0, u), C_2(0, u)\right] = \min(0, 0) = 0,$$

$$C(1, u) = \min\left[C_1(1, u), C_2(1, u)\right] = \min(u, u) = u,$$

$$C(u, 1) = \min\left[C_1(u, 1), C_2(u, 1)\right] = \min(u, u) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \begin{cases} \frac{\partial C_1(u_1,u_2)}{\partial u_1}, & \text{if } C_1(u_1, u_2) < C_2(u_1, u_2), \\ \frac{\partial C_2(u_1,u_2)}{\partial u_1}, & \text{if } C_1(u_1, u_2) \geq C_2(u_1, u_2) \end{cases} \geq 0$$

and

$$\frac{\partial}{\partial u_2} C\left(u_1, u_2\right) = \begin{cases} \frac{\partial C_1\left(u_1, u_2\right)}{\partial u_2}, & \text{if } C_1\left(u_1, u_2\right) < C_2\left(u_1, u_2\right), \\ \frac{\partial C_2\left(u_1, u_2\right)}{\partial u_2}, & \text{if } C_1\left(u_1, u_2\right) \geq C_2\left(u_1, u_2\right) \end{cases} \geq 0,$$

so C is a valid copula.

(4 marks)

UNSEEN

b) for C defined by

$$C\left(u_1, u_2\right) = \max\left[C_1\left(u_1, u_2\right), C_2\left(u_1, u_2\right)\right],$$

we have

$$C\left(u, 0\right) = \max\left[C_1\left(u, 0\right), C_2\left(u, 0\right)\right] = \max(0, 0) = 0,$$

$$C\left(0, u\right) = \max\left[C_1\left(0, u\right), C_2\left(0, u\right)\right] = \max(0, 0) = 0,$$

$$C\left(1, u\right) = \max\left[C_1\left(1, u\right), C_2\left(1, u\right)\right] = \max(u, u) = u,$$

$$C\left(u, 1\right) = \max\left[C_1\left(u, 1\right), C_2\left(u, 1\right)\right] = \max(u, u) = u,$$

$$\frac{\partial}{\partial u_1} C\left(u_1, u_2\right) = \begin{cases} \frac{\partial C_1\left(u_1, u_2\right)}{\partial u_1}, & \text{if } C_1\left(u_1, u_2\right) > C_2\left(u_1, u_2\right), \\ \frac{\partial C_2\left(u_1, u_2\right)}{\partial u_1}, & \text{if } C_1\left(u_1, u_2\right) \leq C_2\left(u_1, u_2\right) \end{cases} \geq 0$$

and

$$\frac{\partial}{\partial u_2} C\left(u_1, u_2\right) = \begin{cases} \frac{\partial C_1\left(u_1, u_2\right)}{\partial u_2}, & \text{if } C_1\left(u_1, u_2\right) > C_2\left(u_1, u_2\right), \\ \frac{\partial C_2\left(u_1, u_2\right)}{\partial u_2}, & \text{if } C_1\left(u_1, u_2\right) \leq C_2\left(u_1, u_2\right) \end{cases} \geq 0,$$

so C is a valid copula.

(4 marks)

UNSEEN

c) for C defined by

$$C\left(u_1, u_2\right) = \sum_{i=1}^{\infty} \alpha_i C_i\left(u_1, u_2\right),$$

where C_i are valid copulas and α_i are non-negative real numbers summing to 1, we have

$$C(u,0) = \sum_{i=1}^{\infty} \alpha_i C_i(u,0) = 0,$$

$$C(0,u) = \sum_{i=1}^{\infty} \alpha_i C_i(0,u) = 0,$$

$$C(u,1) = \sum_{i=1}^{\infty} \alpha_i C_i(u,1) = \sum_{i=1}^{\infty} \alpha_i u = u,$$

$$C(1,u) = \sum_{i=1}^{\infty} \alpha_i C_i(1,u) = \sum_{i=1}^{\infty} \alpha_i u = u,$$

$$\frac{\partial}{\partial u_1} C(u_1,u_2) = \sum_{i=1}^{\infty} \alpha_i \frac{\partial}{\partial u_1} C_i(u_1,u_2) \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1,u_2) = \sum_{i=1}^{\infty} \alpha_i \frac{\partial}{\partial u_2} C_i(u_1,u_2) \geq 0,$$

so C is a valid copula.

(4 marks)

UNSEEN

d) for C defined by

$$C(u_1,u_2) = \prod_{i=1}^{\infty} \left[C_i(u_1,u_2)\right]^{\alpha_i},$$

where C_i are valid copulas and α_i are non-negative real numbers summing to 1, we have

$$C(u,0) = \prod_{i=1}^{\infty} \left[C_i(u,0)\right]^{\alpha_i} = 0,$$

$$C(0,u) = \prod_{i=1}^{\infty} \left[C_i(0,u)\right]^{\alpha_i} = 0,$$

$$C(1,u) = \prod_{i=1}^{\infty} [C_i(1,u)]^{\alpha_i} = \prod_{i=1}^{\infty} u^{\alpha_i} = u^{\alpha_1+\cdots} = 1,$$

$$C(u,1) = \prod_{i=1}^{\infty} [C_i(u,1)]^{\alpha_i} = \prod_{i=1}^{\infty} u^{\alpha_i} = u^{\alpha_1+\cdots} = 1,$$

$$\frac{\partial}{\partial u_1}(u_1,u_2) = \sum_{i=1}^{\infty} \alpha_1 [C_i(u_1,u_2)]^{\alpha_i-1} \frac{\partial C_i(u_1,u_2)}{\partial u_1} \prod_{j=1, j\neq i}^{\infty} [C_j(u_1,u_2)]^{\alpha_j} \geq 0$$

and

$$\frac{\partial}{\partial u_2}(u_1,u_2) = \sum_{i=1}^{\infty} \alpha_1 [C_i(u_1,u_2)]^{\alpha_i-1} \frac{\partial C_i(u_1,u_2)}{\partial u_2} \prod_{j=1, j\neq i}^{\infty} [C_j(u_1,u_2)]^{\alpha_j} \geq 0,$$

so C is a valid copula.

(4 marks)

UNSEEN

Solutions to Question A3

ILO: bivariate extreme value distributions

a) We can write

$$\overline{F}(x,y) = \exp\left\{ -(x+y) \sum_{i=1}^{\infty} \alpha_i A_i \left(\frac{y}{x+y} \right) \right\}$$

for $x > 0$, $y > 0$ and $\alpha_i \geq 0$ sum to one. This is in the form of

$$\overline{F}(x,y) = \exp\left[-(x+y) A \left(\frac{y}{x+y} \right) \right]$$

with

$$A(w) = \sum_{i=1}^{\infty} \alpha_i A_i(w).$$

We now check the conditions for $A(\cdot)$. Clearly,

$$A(0) = \sum_{i=1}^{\infty} \alpha_i A_i(0) = \sum_{i=1}^{\infty} \alpha_i \cdot 1 = 1$$

and

$$A(1) = \sum_{i=1}^{\infty} \alpha_i A_i(1) = \sum_{i=1}^{\infty} \alpha_i \cdot 1 = 1.$$

Also $A(t) \geq 0$ since $A_i(w) \geq 0$ for all w and every k.

Also since each $A_i(w) \leq 1$,

$$A(w) = \sum_{i=1}^{\infty} \alpha_i A_i(w) \leq \sum_{i=1}^{\infty} \alpha_i \cdot 1 = 1.$$

Also since each $A_i(w) \geq \max(w, 1-w)$,

$$A(w) = \sum_{i=1}^{\infty} \alpha_i A_i(w) \geq \sum_{i=1}^{\infty} \alpha_i \cdot \max(w, 1-w) = \max(w, 1-w).$$

Also since each $A_i(w)$ is convex,

$$A^{''}(w) = \sum_{i=1}^{\infty} \alpha_i A_i^{''}(w) > 0.$$

(7 marks)

UNSEEN

b) Note that

$$\overline{F}(x,0) = \exp\left\{-(x+0)\sum_{i=1}^{\infty}\alpha_i A_i\left(0\right)\right\} = \exp(-x)$$

and

$$\overline{F}(0,y) = \exp\left\{-(0+y)\sum_{i=1}^{\infty}\alpha_i A_i\left(0\right)\right\} = \exp(-y).$$

So, the joint cdf is

$$F(x,y) = 1 - \exp(-x) - \exp(-y) + \exp\left\{-(x+y)\sum_{i=1}^{\infty}\alpha_i A_i\left(\frac{y}{x+y}\right)\right\}.$$

(1 marks)

UNSEEN

c) the derivative of joint cdf with respect to x is

$$\frac{\partial F(x,y)}{\partial x} = \exp(-x) + \overline{F}(x,y)\left\{-\sum_{i=1}^{\infty}\alpha_i A_i\left(\frac{y}{x+y}\right) + \frac{y}{x+y}\sum_{i=1}^{\infty}\alpha_i A_i^{'}\left(\frac{y}{x+y}\right)\right\},$$

so the conditional cdf of Y given $X=x$ is

$$F(y \mid x) = 1 + \exp(x)\overline{F}(x,y)\left\{-\sum_{i=1}^{\infty}\alpha_i A_i\left(\frac{y}{x+y}\right) + \frac{y}{x+y}\sum_{i=1}^{\infty}\alpha_i A_i^{'}\left(\frac{y}{x+y}\right)\right\}.$$

(4 marks)

UNSEEN

d) the derivative of joint cdf with respect to y is

$$\frac{\partial F(x,y)}{\partial y} = \exp(-y) + \overline{F}(x,y)\left\{-\sum_{i=1}^{\infty}\alpha_i A_i\left(\frac{y}{x+y}\right) - \frac{x}{x+y}\sum_{i=1}^{\infty}\alpha_i A_i^{'}\left(\frac{y}{x+y}\right)\right\},$$

so the conditional cdf of Y given $X=x$ is

$$F(x \mid y) = 1 + \exp(y)\overline{F}(x,y)\left\{-\sum_{i=1}^{\infty}\alpha_i A_i\left(\frac{y}{x+y}\right) - \frac{x}{x+y}\sum_{i=1}^{\infty}\alpha_i A_i^{'}\left(\frac{y}{x+y}\right)\right\}.$$

(4 marks)

UNSEEN

e) the derivative of joint cdf with respect to x and y is

$$\begin{aligned} f(x,y) &= \overline{F}(x,y)\left\{-\sum_{i=1}^{\infty}\alpha_i A_i\left(\frac{y}{x+y}\right)+\frac{y}{x+y}\sum_{i=1}^{\infty}\alpha_i A_i^{'}\left(\frac{y}{x+y}\right)\right\} \\ &\quad\cdot\left\{-\sum_{i=1}^{\infty}\alpha_i A_i\left(\frac{y}{x+y}\right)-\frac{x}{x+y}\sum_{i=1}^{\infty}\alpha_i A_i^{'}\left(\frac{y}{x+y}\right)\right\} \\ &\quad+\overline{F}(x,y)\frac{xy}{(x+y)^3}\sum_{i=1}^{\infty}\alpha_i A_i^{''}\left(\frac{y}{x+y}\right). \end{aligned}$$

(4 marks)

UNSEEN

Solutions to Question B1

ILO: extreme value distribution of a given univariate distribution

If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cumulative distribution function of a normalized version of M_n converges to G, i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n\left(a_n x + b_n\right) \to G(x)$$

as $n \to \infty$ then G must be of the same type as (cumulative distribution functions G and G^* are of the same type if $G^*(x) = G(ax+b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$\begin{aligned}
I &: \Lambda(x) = \exp\left\{-\exp(-x)\right\}, \qquad x \in \Re; \\
II &: \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\left\{-x^{-\alpha}\right\} & \text{if } x \geq 0 \end{cases} \\
& \quad \text{for some } \alpha > 0; \\
III &: \Psi_\alpha(x) = \begin{cases} \exp\left\{-(-x)^{\alpha}\right\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \\
& \quad \text{for some } \alpha > 0.
\end{aligned}$$

(4 marks)

SEEN

The necessary and sufficient conditions for the three extreme value distributions are:

$$\begin{aligned}
I &: \exists \gamma(t) > 0 \text{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F\left(t + x\gamma(t)\right)}{1 - F(t)} = \exp(-x), \qquad x > 0, \\
II &: w(F) = \infty \text{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \qquad x > 0, \\
III &: w(F) < \infty \text{ and } \lim_{t \downarrow 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} = x^{\alpha}, \qquad x > 0.
\end{aligned}$$

(4 marks)

SEEN

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function say $h(t)$ such that

$$\lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} = e^{-x}$$

for every $x > 0$. But

$$\begin{aligned}
& \lim_{t \to w(F)} \frac{1 - F\left(t + xh(t)\right)}{1 - F(t)} \\
= & \lim_{t \to w(F)} \frac{\left[1 - \left[G\left(t + xh(t)\right)\right]^a \exp\left\{a\left[1 - G\left(t + xh(t)\right)\right]\right\}\right]^b}{\left[1 - \left[G\left(t\right)\right]^a \exp\left\{a\left[1 - G\left(t\right)\right]\right\}\right]^b} \\
= & \lim_{t \to w(F)} \left[\frac{1 - \left[G\left(t + xh(t)\right)\right]^a \exp\left\{a\left[1 - G\left(t + xh(t)\right)\right]\right\}}{1 - \left[G\left(t\right)\right]^a \exp\left\{a\left[1 - G\left(t\right)\right]\right\}}\right]^b \\
= & \lim_{t \to w(G)} \left[\frac{1 - \left[G\left(t + xh(t)\right)\right]^a \exp\left\{a\left[1 - G\left(t + xh(t)\right)\right]\right\}}{1 - \left[G\left(t\right)\right]^a \exp\left\{a\left[1 - G\left(t\right)\right]\right\}}\right]^b \\
= & \lim_{t \to w(G)} \left[\frac{1 - \left[G\left(t + xh(t)\right)\right]^a \exp\left\{a\left[1 - 1\right]\right\}}{1 - \left[G\left(t\right)\right]^a \exp\left\{a\left[1 - 1\right]\right\}}\right]^b \\
= & \lim_{t \to w(G)} \left[\frac{1 - \left[G\left(t + xh(t)\right)\right]^a}{1 - \left[G\left(t\right)\right]^a}\right]^b \\
= & \lim_{t \to w(G)} \left[\frac{1 - \left\{1 - \left[1 - G\left(t + xh(t)\right)\right]\right\}^a}{1 - \left\{1 - \left[1 - G\left(t\right)\right]\right\}^a}\right]^b \\
= & \lim_{t \to w(G)} \left[\frac{1 - \left\{1 - a\left[1 - G\left(t + xh(t)\right)\right]\right\}}{1 - \left\{1 - a\left[1 - G\left(t\right)\right]\right\}}\right]^b \\
= & \lim_{t \to w(G)} \left[\frac{1 - G\left(t + xh(t)\right)}{1 - G\left(t\right)}\right]^b \\
= & \left[\exp(-x)\right]^b \\
= & \exp(-bx)
\end{aligned}$$

for every $x > 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left[-\exp(-bx)\right]$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t \to \infty} \frac{1 - G\left(tx\right)}{1 - G(t)} = x^{-\beta}$$

for every $x > 0$. But

$$\begin{aligned}
& \lim_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} \\
= & \lim_{t\to\infty} \frac{\left[1 - [G(tx)]^a \exp\{a[1 - G(tx)]\}\right]^b}{\left[1 - [G(t)]^a \exp\{a[1 - G(t)]\}\right]^b} \\
= & \lim_{t\to\infty} \left[\frac{1 - [G(tx)]^a \exp\{a[1 - G(tx)]\}}{1 - [G(t)]^a \exp\{a[1 - G(t)]\}}\right]^b \\
= & \lim_{t\to\infty} \left[\frac{1 - [G(tx)]^a \exp\{a[1 - 1]\}}{1 - [G(t)]^a \exp\{a[1 - 1]\}}\right]^b \\
= & \lim_{t\to\infty} \left[\frac{1 - [G(tx)]^a}{1 - [G(t)]^a}\right]^b \\
= & \lim_{t\to\infty} \left[\frac{1 - \{1 - [1 - G(tx)]\}^a}{1 - \{1 - [1 - G(t)]\}^a}\right]^b \\
= & \lim_{t\to\infty} \left[\frac{1 - \{1 - a[1 - G(tx)]\}}{1 - \{1 - a[1 - G(t)]\}}\right]^b \\
= & \lim_{t\to\infty} \left[\frac{1 - G(tx)}{1 - G(t)}\right]^b \\
= & \left[x^{-\beta}\right]^b \\
= & x^{-b\beta}
\end{aligned}$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \le x\right) = \exp\left(-x^{-b\beta}\right)$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t\to 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} = x^{\beta}$$

for every $x > 0$. But

$$
\begin{aligned}
& \lim_{t \to 0} \frac{1 - F\left(t + x h(t)\right)}{1 - F(t)} \\
= & \lim_{t \to 0} \frac{\left[1 - \left[G\left(w(F) - tx\right)\right]^a \exp\left\{a\left[1 - G\left(w(F) - tx\right)\right]\right\}\right]^b}{\left[1 - \left[G\left(w(F) - t\right)\right]^a \exp\left\{a\left[1 - G\left(w(F) - t\right)\right]\right\}\right]^b} \\
= & \lim_{t \to 0} \left[\frac{1 - \left[G\left(w(F) - tx\right)\right]^a \exp\left\{a\left[1 - G\left(w(F) - tx\right)\right]\right\}}{1 - \left[G\left(w(F) - t\right)\right]^a \exp\left\{a\left[1 - G\left(w(F) - t\right)\right]\right\}}\right]^b \\
= & \lim_{t \to 0} \left[\frac{1 - \left[G\left(w(G) - tx\right)\right]^a \exp\left\{a\left[1 - G\left(w(G) - tx\right)\right]\right\}}{1 - \left[G\left(w(G) - t\right)\right]^a \exp\left\{a\left[1 - G\left(w(G) - t\right)\right]\right\}}\right]^b \\
= & \lim_{t \to 0} \left[\frac{1 - \left[G\left(w(G) - tx\right)\right]^a \exp\left\{a\left[1 - 1\right]\right\}}{1 - \left[G\left(w(G) - t\right)\right]^a \exp\left\{a\left[1 - 1\right]\right\}}\right]^b \\
= & \lim_{t \to 0} \left[\frac{1 - \left[G\left(w(G) - tx\right)\right]^a}{1 - \left[G\left(w(G) - t\right)\right]^a}\right]^b \\
= & \lim_{t \to 0} \left[\frac{1 - \left\{1 - \left[1 - G\left(w(G) - tx\right)\right]\right\}^a}{1 - \left\{1 - \left[1 - G\left(w(G) - t\right)\right]\right\}^a}\right]^b \\
= & \lim_{t \to 0} \left[\frac{1 - \left\{1 - a\left[1 - G\left(w(G) - tx\right)\right]\right\}}{1 - \left\{1 - a\left[1 - G\left(w(G) - t\right)\right]\right\}}\right]^b \\
= & \lim_{t \to 0} \left[\frac{1 - G\left(w(G) - tx\right)}{1 - G\left(w(G) - t\right)}\right]^b \\
= & \left[x^\beta\right]^b \\
= & x^{b\beta}
\end{aligned}
$$

for every $x > 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$
\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left(-(-x)^{b\beta}\right)
$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

UNSEEN

Solutions to Question B2

ILO: extreme value distribution of a given univariate distribution

a) Note that $w(F) = \infty$ and

$$
\begin{aligned}
& \lim_{t \to \infty} \frac{1 - F\left(t + x\gamma(t)\right)}{1 - F\left(t\right)} \\
= & \lim_{t \to \infty} \frac{1 - \exp\left[-\exp\left(-t - x\gamma(t)\right)\right]}{1 - \exp\left[-\exp(-t)\right]} \\
= & \lim_{t \to \infty} \frac{1 - \left[1 - \exp\left(-t - x\gamma(t)\right)\right]}{1 - \left[1 - \exp(-t)\right]} \\
= & \lim_{t \to \infty} \frac{\exp\left(-t - x\gamma(t)\right)}{\exp(-t)} \\
= & \lim_{t \to \infty} \exp\left(-x\gamma(t)\right) \\
= & \exp(-x)
\end{aligned}
$$

if $\gamma(t) = 1$. Hence, $F(x)$ belongs to the Gumbel domain of attraction.

(4 marks)

SEEN

b) Note that $w(F) = \infty$ and

$$
\begin{aligned}
& \lim_{t \to \infty} \frac{1 - F\left(tx\right)}{1 - F\left(t\right)} \\
= & \lim_{t \to \infty} \frac{1 - \exp\left(-(tx)^{-\alpha}\right)}{1 - \exp\left(-t^{-\alpha}\right)} \\
= & \lim_{t \to \infty} \frac{1 - \left[1 - (tx)^{-\alpha}\right]}{1 - \left[1 - t^{-\alpha}\right]} \\
= & \lim_{t \to \infty} \frac{(tx)^{-\alpha}}{t^{-\alpha}} \\
& = 7x^{-\alpha}
\end{aligned}
$$

Hence, $F(x)$ belongs to the Fréchet domain of attraction.

c) Note that $w(F)=\infty$ and

$$\begin{aligned}
& \lim_{t\downarrow\infty}\frac{1-F\left(tx\right)}{1-F(t)} \\
= & \lim_{t\downarrow\infty}\frac{xf\left(tx\right)}{f(t)} \\
= & \lim_{t\downarrow\infty}\frac{x\frac{C(tx)^{a-1}}{(1+tx)^{a+b}}}{\frac{Ct^{a-1}}{(1+t)^{a+b}}} \\
= & \lim_{t\downarrow\infty}x^{a}\left(\frac{1+t}{1+tx}\right)^{a+b} \\
= & \lim_{t\downarrow\infty}x^{a}\left(\frac{\frac{1}{t}+1}{\frac{1}{t}+x}\right)^{a+b} \\
= & x^{-b}.
\end{aligned}$$

Hence, $F(x)$ belongs to the Fréchet domain of attraction.

(4 marks)

UNSEEN

d) Note that $w(F)=\infty$ and

$$\begin{aligned}
& \lim_{t\downarrow\infty}\frac{1-F\left(tx\right)}{1-F(t)} \\
= & \lim_{t\downarrow\infty}\frac{xf\left(tx\right)}{f(t)} \\
= & \lim_{t\downarrow\infty}\frac{xC\left(1+at^{2}x^{2}\right)^{-1}\left(1+bt^{2}x^{2}\right)^{-1}}{C\left(1+at^{2}\right)^{-1}\left(1+bt^{2}\right)^{-1}} \\
= & \lim_{t\downarrow\infty}x\frac{1+at^{2}}{1+at^{2}x^{2}}\frac{1+bt^{2}}{1+bt^{2}x^{2}} \\
= & \lim_{t\downarrow\infty}x\frac{t^{-2}+a}{t^{-2}+ax^{2}}\frac{t^{-2}+b}{t^{-2}+bx^{2}} \\
= & x^{-3}.
\end{aligned}$$

Hence, $F(x)$ belongs to the Fréchet domain of attraction.

(4 marks)

UNSEEN

e) Note that $w(F) = N$ and

$$\begin{aligned}
\frac{\Pr(X = w(F))}{1 - F(w(F) - 1)} &= \frac{\Pr(X = N)}{1 - F(N-1)} \\
&= \frac{\Pr(X = N)}{1 - \Pr(X = 1) - \Pr(X = 2) - \cdots - \Pr(X = N-1)} \\
&= \frac{\frac{1}{N}}{1 - \frac{1}{N} - \frac{1}{N} - \cdots - \frac{1}{N}} \\
&= \frac{\frac{1}{N}}{1 - \frac{N-1}{N}} \\
&= 1.
\end{aligned}$$

Hence, there can be no non-degenerate limit.

(4 marks)

UNSEEN

Solutions to Question B3

ILO: estimates of financial risk measures and total portfolio loss

(a) If X is an absolutely continuous random variable with cumulative distribution function $F(\cdot)$ then

$$\mathrm{VaR}_p(X) = F^{-1}(p)$$

and

$$\mathrm{ES}_p(X) = \frac{1}{p}\int_0^p F^{-1}(v)dv.$$

(2 marks)

SEEN

(b) (i) The probability density function of S is

$$\begin{aligned} f_S(s) &= \int_0^s f_{X,Y}(x, s-x)dx \\ &= (a+1)(a+2)\int_0^s (1-s)^a dx \\ &= (a+1)(a+2)s(1-s)^a. \end{aligned}$$

for $0 < s < 1$.

(2 marks)

UNSEEN

(b) (ii) The cumulative distribution function of S is

$$\begin{aligned} F_S(s) &= \int_0^s f_S(t)dt \\ &= (a+1)(a+2)\int_0^s t(1-t)^a dt \\ &= (a+1)(a+2)\int_0^s \left[1-(1-t)\right](1-t)^a dt \\ &= (a+1)(a+2)\int_0^s \left[(1-t)^a - (1-t)^{a+1}\right] dt \\ &= (a+1)(a+2)\left[-\frac{(1-t)^{a+1}}{a+1} + \frac{(1-t)^{a+2}}{a+2}\right]_0^s \\ &= 1-(a+2)(1-s)^{a+1} + (a+1)(1-s)^{a+2} \end{aligned}$$

for $0 < s < 1$.

(4 marks)

UNSEEN

(b) (iii) The value at risk of S is the solution of

$$F_S(s) = u$$

which is equivalent to

$$(a+2)(1-s)^{a+1} - (a+1)(1-s)^{a+2} = 1-u.$$

(2 marks)

UNSEEN

(b) (iv) The expected shortfall of S is

$$\begin{aligned}
\mathrm{ES}_p(S) &= \frac{1}{p}\int_0^p F_S^{-1}(u)du \\
&= \frac{1}{p}\int_0^{F_S^{-1}(p)} s f_S(s)ds \\
&= \frac{(a+1)(a+2)}{p}\int_0^{F_S^{-1}(p)} s^2(1-s)^a ds \\
&= \frac{(a+1)(a+2)}{p}\int_0^{F_S^{-1}(p)} \left[1-(1-s)\right]^2 (1-s)^a ds \\
&= \frac{(a+1)(a+2)}{p}\int_0^{F_S^{-1}(p)} \left[(1-s)^a - 2(1-s)^{a+1} + (1-s)^{a+2}\right] ds \\
&= \frac{(a+1)(a+2)}{p}\left[-\frac{(1-s)^{a+1}}{a+1} + 2\frac{(1-s)^{a+1}}{a+2} - \frac{(1-s)^{a+3}}{a+3}\right]_0^{F_S^{-1}(p)} \\
&= \frac{(a+1)(a+2)}{p}\left[\frac{1}{a+1} - \frac{2}{a+2} + \frac{1}{a+3}\right. \\
&\qquad \left. - \frac{\left(1-F_S^{-1}(p)\right)^{a+1}}{a+1} + 2\frac{\left(1-F_S^{-1}(p)\right)^{a+1}}{a+2} - \frac{\left(1-F_S^{-1}(p)\right)^{a+3}}{a+3}\right].
\end{aligned}$$

(5 marks)

UNSEEN

(b) (v) The likelihood function is

$$\begin{aligned} L(a) &= \prod_{i=1}^{n} \left[(a+1)(a+2)s_i \left(1-s_i\right)^a\right] \\ &= (a+1)^n (a+2)^n \left(\prod_{i=1}^{n} s_i\right) \left[\prod_{i=1}^{n} \left(1-s_i\right)\right]^a \end{aligned}$$

The log-likelihood function is

$$\log L(a) = n \log(a+1) + n \log(a+2) + \sum_{i=1}^{n} \log s_i + a \sum_{i=1}^{n} \log\left(1-s_i\right).$$

The derivative with respect to a is

$$\frac{d \log L(a)}{a} = \frac{n}{a+1} + \frac{n}{a+2} + \sum_{i=1}^{n} \log\left(1-s_i\right).$$

Setting this to zero, we obtain

$$\frac{2a+3}{(a+1)(a+2)} = -\frac{1}{n} \sum_{i=1}^{n} \log\left(1-s_i\right),$$

which can be rewritten as

$$Aa^2 + Ba + C = 0,$$

where

$$\begin{aligned} A &= -\frac{1}{n} \sum_{i=1}^{n} \log\left(1-s_i\right), \\ B &= -\frac{3}{n} \sum_{i=1}^{n} \log\left(1-s_i\right) - 2, \\ C &= -\frac{2}{n} \sum_{i=1}^{n} \log\left(1-s_i\right) - 3. \end{aligned}$$

The valid root is $\left(-B + \sqrt{B^2 - 4AC}\right)/(2A)$. This is a maximum likelihood estimator since

$$\frac{d^2 \log L(a)}{a^2} = -\frac{n}{(a+1)^2} - \frac{n}{(a+2)^2} < 0.$$

(5 marks)

UNSEEN

Solutions to Question B4

ILO: estimates of financial risk measures and minimum portfolio loss

(i) The cumulative distribution function of $U = \min(X, Y)$ is

$$\begin{aligned} F_U(u) &= \Pr(U \leq u) \\ &= 1 - \Pr(U > u) \\ &= 1 - \Pr\left(\min(X, Y) > u\right) \\ &= 1 - \Pr\left(X > u, Y > u\right) \\ &= 1 - \overline{F}(u, u) \\ &= 1 - \left(\frac{K}{u}\right)^a \end{aligned}$$

for $u > K$.

(2 marks)

UNSEEN

(ii) The probability density function of U is

$$f_U(u) = \frac{aK^a}{u^{a+1}}$$

for $u > K$.

(1 marks)

UNSEEN

(iii) The mth moment of U is

$$\begin{aligned} E\left(U^m\right) &= \int_K^\infty u^m \frac{aK^a}{u^{a+1}} du \\ &= aK^a \int_K^\infty u^{m-a-1} du \\ &= aK^a \left[\frac{u^{m-a}}{m-a}\right]_K^\infty \\ &= aK^a \left[0 - \frac{K^{m-a}}{m-a}\right]. \\ &= \frac{aK^m}{a-m}. \end{aligned}$$

(4 marks)

UNSEEN

(iv) The mean of U is

$$E\left(U\right) = \frac{aK}{a-1}.$$

(1 marks)

UNSEEN

(v) The variance of U is

$$Var(U) = E\left(U^2\right) - \left[E(U)\right]^2 = \frac{aK^2}{a-2} - \left(\frac{aK}{a-1}\right)^2.$$

(1 marks)

UNSEEN

(vi) Setting

$$F_U(u) = p$$

gives

$$1 - \left(\frac{K}{u}\right)^a = p$$

which implies

$$u = K(1-p)^{-1/a}.$$

Hence, $\mathrm{VaR}_p(U) = K(1-p)^{-1/a}$.

(2 marks)

UNSEEN

(vii) The expected shortfall is

$$\begin{aligned}
\mathrm{ES}_p(U) &= \frac{K}{p}\int_0^p (1-t)^{-1/a} dt \\
&= \frac{K}{p}\left[\frac{(1-t)^{1-1/a}}{1/a-1}\right]_0^p \\
&= \frac{K}{p(1/a-1)}\left[(1-p)^{1-1/a} - 1\right].
\end{aligned}$$

(4 marks)

UNSEEN

(viii) The likelihood function is

$$\begin{aligned} L(K,a) &= \prod_{i=1}^{n}\left[\frac{aK^a}{u_i^{a+1}} I\left\{u_i > K\right\}\right] \\ &= a^n K^{na}\left(\prod_{i=1}^{n} u_i\right)^{-a-1}\left[\prod_{i=1}^{n} I\left\{u_i > K\right\}\right] \\ &= a^n K^{na}\left(\prod_{i=1}^{n} u_i\right)^{-a-1} I\left\{\min\left(u_1,\ldots,u_n\right) > K\right\} \end{aligned}$$

Note that $L(K,a)$ is an increasing function of K over $(-\infty, \min\left(u_1,\ldots,u_n\right))$. Hence, the maximum likelihood estimator of K is $\widehat{K} = \min\left(u_1,\ldots,u_n\right)$. The log-likelihood function is

$$\log L(K,a) = n\log a + na\log K - (a+1)\sum_{i=1}^{n}\log u_i.$$

The partial derivative with respect to a is

$$\frac{\partial \log L(K,a)}{\partial a} = \frac{n}{a} + n\log K - \sum_{i=1}^{n}\log u_i.$$

Setting this to zero, we obtain the maximum likelihood estimator of a as

$$\widehat{a} = n\left[-n\log K + \sum_{i=1}^{n}\log u_i\right]^{-1}$$

(4 marks)

UNSEEN

(ix) The maximum likelihood estimator of Value at Risk and Expected Shortfall are

$$\widehat{\mathrm{VaR}}_p(U) = \widehat{K}(1-p)^{-1/\widehat{a}}$$

and

$$\widehat{\mathrm{ES}}_p(U) = \frac{\widehat{K}}{p(1/\widehat{a}-1)}\left[(1-p)^{1-1/\widehat{a}} - 1\right].$$

(1 marks)

Solutions to Question B5

ILO: estimates of financial risk measures and maximum portfolio loss

(i) The cumulative distribution function of $V = \max(X, Y)$ is

$$\begin{aligned} F_V(v) &= \Pr(V \leq v) \\ &= \Pr\left(\max(X, Y) \leq v\right) \\ &= \Pr\left(X \leq v, Y \leq v\right) \\ &= F_{X,Y}(v, v) \\ &= \left(\frac{v}{K}\right)^a \end{aligned}$$

for $0 \leq v \leq K$.

(2 marks)

UNSEEN

(ii) The probability density function of V is

$$f_V(v) = \frac{av^{a-1}}{K^a}$$

for $0 \leq v \leq K$.

(1 marks)

UNSEEN

(iii) The mth moment of U is

$$\begin{aligned} E\left(V^m\right) &= \int_0^K v^m \frac{av^{a-1}}{K^a} dv \\ &= \frac{a}{K^a} \int_0^K v^{m+a-1} dv \\ &= \frac{a}{K^a} \left[\frac{v^{m+a}}{m+a}\right]_0^K \\ &= \frac{a}{K^a} \left[\frac{K^{m+a}}{m+a} - 0\right]. \\ &= \frac{aK^m}{m+a}. \end{aligned}$$

(4 marks)

UNSEEN

(iv) The mean of V is

$$E\left(V\right)=\frac{aK}{1+a}.$$

(1 marks)

UNSEEN

(v) The variance of V is

$$Var(V)=E\left(V^2\right)-\left[E(V)\right]^2=\frac{aK^2}{2+a}-\left(\frac{aK}{1+a}\right)^2.$$

(1 marks)

UNSEEN

(vi) Setting

$$F_V(v)=p$$

gives

$$\left(\frac{v}{K}\right)^a=p$$

which implies

$$v=Kp^{1/a}.$$

Hence, $\mathrm{VaR}_p(V)=Kp^{1/a}$.

(2 marks)

UNSEEN

(vii) The expected shortfall is

$$\begin{aligned}\mathrm{ES}_p(V) &= \frac{K}{p}\int_0^p t^{1/a}dt \\ &= \frac{K}{p}\left[\frac{t^{1+1/a}}{1+1/a}\right]_0^p \\ &= \frac{Kp^{1/a}}{1+1/a}.\end{aligned}$$

(4 marks)

UNSEEN

(viii) The likelihood function is

$$\begin{aligned} L(K,a) &= \prod_{i=1}^{n}\left[\frac{av_i^{a-1}}{K^a} I\left\{v_i \le K\right\}\right] \\ &= \frac{a^n}{K^{na}}\left(\prod_{i=1}^{n} v_i\right)^{a-1}\left[\prod_{i=1}^{n} I\left\{v_i \le K\right\}\right] \\ &= \frac{a^n}{K^{na}}\left(\prod_{i=1}^{n} v_i\right)^{a-1} I\left\{\max\left(u_1,\ldots,u_n\right) \le K\right\} \end{aligned}$$

Note that $L(K,a)$ is a decreasing function of K over $[\max\left(u_1,\ldots,u_n\right),\infty)$. Hence, the maximum likelihood estimator of K is $\widehat{K} = \max\left(u_1,\ldots,u_n\right)$. The log-likelihood function is

$$\log L(K,a) = n\log a - na\log K + (a-1)\sum_{i=1}^{n}\log v_i.$$

The partial derivative with respect to a is

$$\frac{\partial \log L(K,a)}{\partial a} = \frac{n}{a} - n\log K + \sum_{i=1}^{n}\log v_i.$$

Setting this to zero, we obtain the maximum likelihood estimator of a as

$$\widehat{a} = n\left[n\log K - \sum_{i=1}^{n}\log u_i\right]^{-1}$$

(4 marks)

UNSEEN

(ix) The maximum likelihood estimator of Value at Risk and Expected Shortfall are

$$\widehat{\mathrm{VaR}}_p(U) = \widehat{K}p^{1/\widehat{a}}$$

and

$$\widehat{\mathrm{ES}}_p(U) = \frac{\widehat{K}p^{1/\widehat{a}}}{1+1/\widehat{a}}.$$

(1 marks)