SOLUTIONS TO MATH68181 EXTREME VALUES AND FINANCIAL RISK EXAM

Solutions to Question A1

ILO: bivariate extreme value distributions

a) The joint cumulative distribution function of X and Y is

$$\begin{aligned}
F_{X,Y}(x,y) &= \int_0^x \int_0^y f_{X,Y}(u,v)dvdu \\
&= \int_0^x \int_0^y (u+v)dvdu \\
&= \int_0^x \left[uv + \frac{v^2}{2}\right]_0^y du \\
&= \int_0^x \left[uy + \frac{y^2}{2} - 0\right] du \\
&= y\int_0^x udu + \frac{y^2}{2}\int_0^x 1du \\
&= y\frac{x^2}{2} + \frac{y^2}{2}x \\
&= \frac{1}{2}xy(x+y).
\end{aligned}$$

(3 marks)

UNSEEN

b) The marginal cumulative distribution functions of X and Y are

$$F_X(x) = F_{X,Y}(x,1) = \frac{1}{2}x(x+1)$$

and

$$F_Y(y) = F_{X,Y}(1,y) = \frac{1}{2}y(y+1).$$

(2 marks)

UNSEEN

c) First note that $w(F_X) = 1$. F_X belongs to the Weibull max domain of attraction since

$$\begin{aligned}
\lim_{t \to 0} \frac{1 - F_X(1-tx)}{1 - F_X(1-t)} &= \lim_{t \to 0} \frac{1 - \frac{1}{2}(1-tx)(2-tx)}{1 - \frac{1}{2}(1-t)(2-t)} \\
&= \lim_{t \to 0} \frac{1 - \left(1 - \frac{3}{2}tx + \frac{1}{2}t^2x^2\right)}{1 - \left(1 - \frac{3}{2}t + \frac{1}{2}t^2\right)} \\
&= \lim_{t \to 0} \frac{\frac{3}{2}tx - \frac{1}{2}t^2x^2}{\frac{3}{2}t - \frac{1}{2}t^2} \\
&= \lim_{t \to 0} \frac{\frac{3}{2}x - \frac{1}{2}tx^2}{\frac{3}{2} - \frac{1}{2}t} \\
&= \lim_{t \to 0} \frac{\frac{3}{2}x}{\frac{3}{2}} \\
&= x.
\end{aligned}$$

(2 marks)

UNSEEN

d) First note that $w(F_Y) = 1$. F_Y belongs to the Weibull max domain of attraction since

$$\begin{aligned}
\lim_{t \to 0} \frac{1 - F_Y(1-ty)}{1 - F_Y(1-t)} &= \lim_{t \to 0} \frac{1 - \frac{1}{2}(1-ty)(2-ty)}{1 - \frac{1}{2}(1-t)(2-t)} \\
&= \lim_{t \to 0} \frac{1 - \left(1 - \frac{3}{2}ty + \frac{1}{2}t^2y^2\right)}{1 - \left(1 - \frac{3}{2}t + \frac{1}{2}t^2\right)} \\
&= \lim_{t \to 0} \frac{\frac{3}{2}ty - \frac{1}{2}t^2y^2}{\frac{3}{2}t - \frac{1}{2}t^2} \\
&= \lim_{t \to 0} \frac{\frac{3}{2}y - \frac{1}{2}ty^2}{\frac{3}{2} - \frac{1}{2}t} \\
&= \lim_{t \to 0} \frac{\frac{3}{2}y}{\frac{3}{2}} \\
&= y.
\end{aligned}$$

(2 marks)

UNSEEN

e) Use the formulas $a_n = w(F_X) - F_X^{-1}\left(1 - \frac{1}{n}\right)$ and $b_n = w(F_X)$. Inverting

$$F_X(x) = \frac{1}{2}x(x+1) = p$$

gives

$$x^2 + x - 2p = 0.$$

The valid root of this equation is

$$x = \frac{-1 + \sqrt{1 + 8p}}{2}.$$

So,

$$F_X^{-1}\left(1 - \frac{1}{n}\right) = \frac{-1 + \sqrt{9 - \frac{8}{n}}}{2}.$$

Hence,

$$a_n = 1 - \frac{-1 + \sqrt{9 - \frac{8}{n}}}{2}, \qquad b_n = 1.$$

(2 marks)

UNSEEN

f) Use the formulas $c_n = w\left(F_Y\right) - F_Y^{-1}\left(1 - \frac{1}{n}\right)$ and $d_n = w\left(F_Y\right)$. Inverting

$$F_Y(y) = \frac{1}{2}y(y+1) = p$$

gives

$$y^2 + y - 2p = 0.$$

The valid root of this equation is

$$y = \frac{-1 + \sqrt{1 + 8p}}{2}.$$

So,

$$F_Y^{-1}\left(1 - \frac{1}{n}\right) = \frac{-1 + \sqrt{9 - \frac{8}{n}}}{2}.$$

Hence,

$$c_n = 1 - \frac{-1 + \sqrt{9 - \frac{8}{n}}}{2}, \qquad d_n = 1.$$

(2 marks)

UNSEEN

g) The limiting cumulative distribution function is

$$
\begin{aligned}
& \lim_{n\to\infty} F_{X,Y}^n\left(a_n x + b_n, c_n y + d_n\right) \\
= & \lim_{n\to\infty} \frac{1}{2^n}\left(a_n x + b_n\right)^n \left(c_n y + d_n\right)^n \left(a_n x + b_n + c_n y + d_n\right)^n \\
= & \lim_{n\to\infty} \frac{1}{2^n}\left[\left(1 - \frac{-1 + \sqrt{9 - \frac{8}{n}}}{2}\right) x + 1\right]^n \left[\left(1 - \frac{-1 + \sqrt{9 - \frac{8}{n}}}{2}\right) y + 1\right]^n \\
& \cdot \left[\left(1 - \frac{-1 + \sqrt{9 - \frac{8}{n}}}{2}\right)(x+y) + 2\right]^n \\
= & \lim_{n\to\infty} \left[\left(1 - \frac{-1 + \sqrt{9 - \frac{8}{n}}}{2}\right) x + 1\right]^n \left[\left(1 - \frac{-1 + \sqrt{9 - \frac{8}{n}}}{2}\right) y + 1\right]^n \\
& \cdot \left[\left(\frac{1}{2} - \frac{-1 + \sqrt{9 - \frac{8}{n}}}{4}\right)(x+y) + 1\right]^n \\
= & \lim_{n\to\infty} \left[\left(1 - \frac{-1 + 3\sqrt{1 - \frac{8}{9n}}}{2}\right) x + 1\right]^n \left[\left(1 - \frac{-1 + 3\sqrt{1 - \frac{8}{9n}}}{2}\right) y + 1\right]^n \\
& \cdot \left[\left(\frac{1}{2} - \frac{-1 + 3\sqrt{1 - \frac{8}{9n}}}{4}\right)(x+y) + 1\right]^n \\
= & \lim_{n\to\infty} \left[\left(1 - \frac{-1 + 3\left(1 - \frac{4}{9n} + \cdots\right)}{2}\right) x + 1\right]^n \left[\left(1 - \frac{-1 + 3\left(1 - \frac{4}{9n} + \cdots\right)}{2}\right) y + 1\right]^n \\
& \cdot \left[\left(\frac{1}{2} - \frac{-1 + 3\left(1 - \frac{4}{9n} + \cdots\right)}{4}\right)(x+y) + 1\right]^n \\
= & \lim_{n\to\infty} \left[\frac{2}{3n} x + 1\right]^n \left[\frac{2}{3n} y + 1\right]^n \left[\frac{1}{3n}(x+y) + 1\right]^n \\
= & \exp\left(\frac{2x}{3}\right) \exp\left(\frac{2y}{3} y\right) \exp\left(\frac{x+y}{3}\right) \\
= & \exp\left(x+y\right).
\end{aligned}
$$

(5 marks)

UNSEEN

h) Yes, the extremes are completely independent since

$$\begin{aligned}
\lim_{n\to\infty} F^n_{X,Y}\left(a_n x + b_n, c_n y + d_n\right) &= \exp\left(x+y\right) \\
&= \exp\left(x\right)\exp\left(y\right) \\
&= \lim_{n\to\infty} F^n_X\left(a_n x + b_n\right) \lim_{n\to\infty} F^n_Y\left(c_n y + d_n\right).
\end{aligned}$$

(2 marks)

UNSEEN

Solutions to Question A2

ILO: checking a function is a copula

$C(u_1, u_2)$ is a valid copula if

$$C(u, 0) = 0,$$

$$C(0, u) = 0,$$

$$C(1, u) = u,$$

$$C(u, 1) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) \geq 0.$$

(4 marks)

SEEN

a) for the copula defined by $C(u_1, u_2) = \min(u_1, u_2)$, we have

$$C(u, 0) = \min(u, 0) = 0,$$

$$C(0, u) = \min(0, u) = 0,$$

$$C(1, u) = \min(1, u) = u,$$

$$C(u, 1) = \min(u, 1) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \begin{cases} 1, & \text{if } u_1 \leq u_2, \\ 0, & \text{if } u_1 > u_2, \end{cases}$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \begin{cases} 0, & \text{if } u_1 \leq u_2, \\ 1, & \text{if } u_1 > u_2, \end{cases}$$

so C is a valid copula.

(4 marks)

UNSEEN

b) for the copula defined by $C(u_1, u_2) = u_1 u_2 \exp[-\theta \log u_1 \log u_2]$, we have

$$C(u, 0) = u \cdot 0 \cdot \exp[-\theta \log 0 \log u] = 0,$$

$$C(0, u) = 0 \cdot u \exp[-\theta \log u \log 0] = 0,$$

$$C(1, u) = 1 \cdot u \exp[-\theta \log 1 \log u] = u,$$

$$C(u, 1) = u \cdot 1 \exp[-\theta \log u \log 1] = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = u_2 (1 - \theta \log u_2) \exp[-\theta \log u_1 \log u_2] \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = u_1 (1 - \theta \log u_1) \exp[-\theta \log u_1 \log u_2] \geq 0,$$

so C is a valid copula.

(4 marks)

UNSEEN

c) for the Farlie-Gumbel-Morgenstern copula defined by

$$C(u_1, u_2) = u_1 u_2 [1 + \phi (1 - u_1)(1 - u_2)],$$

we have

$$C(u, 0) = u \cdot 0 [1 + \phi (1 - u)(1 - 0)] = 0,$$

$$C(0, u) = 0 \cdot u [1 + \phi (1 - 0)(1 - u)] = 0,$$

$$C(u, 1) = u \cdot 1 [1 + \phi (1 - u)(1 - 1)] = u,$$

$$C(1, u) = 1 \cdot u [1 + \phi (1 - 1)(1 - u)] = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = u_2 [1 + \phi (1 - u_1)(1 - u_2)] - u_1 u_2 \phi (1 - u_2) \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = u_1 [1 + \phi (1 - u_1)(1 - u_2)] - u_1 u_2 \phi (1 - u_1) \geq 0$$

so C is a valid copula.

(4 marks)

UNSEEN

d) for the Burr copula defined by $C(u_1, u_2) = u_1 + u_2 - 1 + \left[(1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1\right]^{-\alpha}$, we have

$$C(u, 0) = u + 0 - 1 + \left[(1 - u)^{-1/\alpha} + (1 - 0)^{-1/\alpha} - 1\right]^{-\alpha} = 0,$$

$$C(0, u) = 0 + u - 1 + \left[(1 - 0)^{-1/\alpha} + (1 - u)^{-1/\alpha} - 1\right]^{-\alpha} = 0,$$

$$C(u, 1) = u + 1 - 1 + \left[(1 - u)^{-1/\alpha} + (1 - 1)^{-1/\alpha} - 1\right]^{-\alpha} = 1,$$

$$C(1, u) = 1 + u - 1 + \left[(1 - 1)^{-1/\alpha} + (1 - u)^{-1/\alpha} - 1\right]^{-\alpha} = 1,$$

$$\begin{aligned} \frac{\partial}{\partial u_1} C(u_1, u_2) &= 1 - (1 - u_1)^{-1/\alpha - 1} \left[(1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1\right]^{-\alpha - 1} \\ &= 1 - \left[\frac{(1 - u_1)^{-1/\alpha}}{(1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1}\right]^{\alpha + 1} \geq 0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial u_2} C(u_1, u_2) &= 1 - (1 - u_2)^{-1/\alpha - 1} \left[(1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1\right]^{-\alpha - 1} \\ &= 1 - \left[\frac{(1 - u_2)^{-1/\alpha}}{(1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1}\right]^{\alpha + 1} \geq 0, \end{aligned}$$

so C is a valid copula.

(4 marks)

UNSEEN

Solutions to Question A3

ILO: bivariate extreme value distributions

a) We can write

$$\overline{F}(x,y) = \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right] = \exp\left\{-(x+y)\left[\theta\frac{y^2}{(x+y)^2} - \theta\frac{y}{x+y} + 1\right]\right\}.$$

This is in the form of

$$\overline{F}(x,y) = \exp\left[-(x+y)A\left(\frac{y}{x+y}\right)\right]$$

with $A(t) = \theta t^2 - \theta t + 1$.

We now check the conditions for $A(\cdot)$. Clearly, $A(0) = 1$ and $A(1) = 1$.

Also $A(t) \geq 0$ since

$$\begin{aligned} & \theta t^2 - \theta t + 1 \geq 0 \\ \Leftrightarrow\ & \theta\left(t^2 - t\right) + 1 \geq 0 \\ \Leftrightarrow\ & \theta(t - 1/2)^2 + 1 - \theta/4 \geq 0, \end{aligned}$$

which always holds.

Also $A(t) \leq 1$ since

$$\begin{aligned} & \theta t^2 - \theta t + 1 \leq 1 \\ \Leftrightarrow\ & \theta t^2 - \theta t \leq 0 \\ \Leftrightarrow\ & \theta t(t-1) \leq 0, \end{aligned}$$

which always holds.

Also $A(t) \geq t$ since

$$\begin{aligned} & \theta t^2 - \theta t + 1 \geq t \\ \Leftrightarrow\ & \theta t^2 - (\theta + 1)t + 1 \geq 0 \\ \Leftrightarrow\ & (1 - \theta t)(1 - t) \geq 0, \end{aligned}$$

which always holds.

Also $A(t) \geq 1 - t$ since

$$\begin{aligned} & \theta t^2 - \theta t + 1 \geq 1 - t \\ \Leftrightarrow\ & \theta t^2 + (1 - \theta)t \geq 0 \\ \Leftrightarrow\ & (\theta t - \theta + 1)t \geq 0, \end{aligned}$$

which always holds.

$A^{'}(t) = 2\theta t - \theta$ and $A^{''}(t) = 2\theta > 0$, so $A(\cdot)$ is convex.

(6 marks)

b) the joint cumulative distribution function is

$$F(x, y) = 1 - \exp(-x) - \exp(-y) + \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right].$$

(2 marks)

c) the derivative of joint cumulative distribution function with respect to x is

$$\frac{\partial F(x, y)}{\partial x} = \exp(-x) + \left[\frac{\theta y^2}{(x+y)^2} - 1\right] \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right],$$

so the conditional cumulative distribution function if Y given $X = x$ is

$$F(y|x) = 1 + \left[\frac{\theta y^2}{(x+y)^2} - 1\right] \exp\left[-\frac{\theta y^2}{x+y} + \theta y - y\right].$$

(4 marks)

d) the derivative of joint cumulative distribution function with respect to y is

$$\frac{\partial F(x, y)}{\partial y} = \exp(-y) + \left[\frac{\theta y^2}{(x+y)^2} - \frac{2\theta y}{x+y} + \theta - 1\right] \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right],$$

so the conditional cumulative distribution function if X given $Y = y$ is

$$F(x|y) = 1 + \left[\frac{\theta y^2}{(x+y)^2} - \frac{2\theta y}{x+y} + \theta - 1\right] \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x\right].$$

(4 marks)

e) the derivative of joint cumulative distribution function with respect to x and y is

$$\begin{aligned} f(x, y) &= \frac{\partial F(x, y)}{\partial x \partial y} \\ &= \left\{\left[\frac{\theta y^2}{(x+y)^2} - 1\right]\left[\frac{\theta y^2}{(x+y)^2} - \frac{2\theta y}{x+y} + \theta - 1\right] + \left[\frac{2\theta y}{(x+y)^2} - \frac{2\theta y^2}{(x+y)^3}\right]\right\} \\ &\quad \times \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right]. \end{aligned}$$

(4 marks)

Solutions to Question B1

ILO: extreme value distribution of a given univariate distribution

If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cumulative distribution function of a normalized version of M_n converges to G, i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n\left(a_n x + b_n\right) \to G(x)$$

as $n \to \infty$ then G must be of the same type as (cumulative distribution functions G and G^* are of the same type if $G^*(x) = G(ax+b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$\begin{aligned}
I &: \Lambda(x) = \exp\left\{-\exp(-x)\right\}, \qquad x \in \Re; \\
II &: \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\left\{-x^{-\alpha}\right\} & \text{if } x \geq 0 \end{cases} \\
& \text{for some } \alpha > 0; \\
III &: \Psi_\alpha(x) = \begin{cases} \exp\left\{-(-x)^{\alpha}\right\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \\
& \text{for some } \alpha > 0.
\end{aligned}$$

(4 marks)

SEEN

The necessary and sufficient conditions for the three extreme value distributions are:

$$\begin{aligned}
I &: \exists \gamma(t) > 0 \text{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F\left(t + x\gamma(t)\right)}{1 - F(t)} = \exp(-x), \qquad x > 0, \\
II &: w(F) = \infty \text{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \qquad x > 0, \\
III &: w(F) < \infty \text{ and } \lim_{t \downarrow 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} = x^{\alpha}, \qquad x > 0.
\end{aligned}$$

(4 marks)

SEEN

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function say $h(t)$ such that

$$\lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} = e^{-x}$$

for every $x > 0$. But

$$
\begin{aligned}
& \lim_{t\to w(F)} \frac{1-F\left(t+xh(t)\right)}{1-F(t)} \\
= & \lim_{t\to w(F)} \frac{1-\frac{aG(t+xh(t))[1+b-bG(t+xh(t))]}{1-(1-a)G(t+xh(t))[1+b-bG(t+xh(t))]}}{1-\frac{aG(t)[1+b-bG(t)]}{1-(1-a)G(t)[1+b-bG(t)]}} \\
= & \lim_{t\to w(G)} \frac{1-\frac{aG(t+xh(t))[1+b-bG(t+xh(t))]}{1-(1-a)G(t+xh(t))[1+b-bG(t+xh(t))]}}{1-\frac{aG(t)[1+b-bG(t)]}{1-(1-a)G(t)[1+b-bG(t)]}} \\
= & \lim_{t\to w(G)} \frac{1-\frac{aG(t+xh(t))[1+b-b]}{1-(1-a)G(t+xh(t))[1+b-b]}}{1-\frac{aG(t)}{1-(1-a)G(t)[1+b-b]}} \\
= & \lim_{t\to w(G)} \frac{1-\frac{aG(t+xh(t))}{1-(1-a)[1+b-b]}}{1-\frac{aG(t)}{1-(1-a)[1+b-b]}} \\
= & \lim_{t\to w(G)} \frac{1-\frac{aG(t+xh(t))}{1-(1-a)}}{1-\frac{aG(t)}{1-(1-a)}} \\
= & \lim_{t\to w(G)} \frac{1-G\left(t+xh(t)\right)}{1-G(t)} \\
= & \exp(-x)
\end{aligned}
$$

for every $x > 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left[-\exp(-x)\right]$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t\to\infty} \frac{1-G\left(tx\right)}{1-G(t)} = x^{-\beta}$$

for every $x > 0$. But

$$
\begin{aligned}
& \lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} \\
= & \lim_{t\to\infty} \frac{1-\frac{aG(tx)[1+b-bG(tx)]}{1-(1-a)G(tx)[1+b-bG(tx)]}}{1-\frac{aG(t)[1+b-bG(t)]}{1-(1-a)G(t)[1+b-bG(t)]}} \\
= & \lim_{t\to\infty} \frac{1-\frac{aG(tx)[1+b-b]}{1-(1-a)G(tx)[1+b-b]}}{1-\frac{aG(t)}{1-(1-a)G(t)[1+b-b]}} \\
= & \lim_{t\to\infty} \frac{1-\frac{aG(tx)}{1-(1-a)[1+b-b]}}{1-\frac{aG(t)}{1-(1-a)[1+b-b]}} \\
= & \lim_{t\to\infty} \frac{1-\frac{aG(tx)}{1-(1-a)}}{1-\frac{aG(t)}{1-(1-a)}} \\
= & \lim_{t\to\infty} \frac{1-G(tx)}{1-G(t)} \\
= & \; x^{-\beta}
\end{aligned}
$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left(-x^{-\beta}\right)$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t\to 0} \frac{1-G(w(G)-tx)}{1-G(w(G)-t)} = x^{\beta}$$

for every $x > 0$. But

$$
\begin{aligned}
& \lim_{t \to 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F(w(F) - t)} \\
= \; & \lim_{t \to 0} \frac{1 - \frac{aG(w(F)-tx)[1+b-bG(w(F)-tx)]}{1-(1-a)G(w(F)-tx)[1+b-bG(w(F)-tx)]}}{1 - \frac{aG(w(F)-t)[1+b-bG(w(F)-t)]}{1-(1-a)G(w(F)-t)[1+b-bG(w(F)-t)]}} \\
= \; & \lim_{t \to 0} \frac{1 - \frac{aG(w(G)-tx)[1+b-bG(w(G)-tx)]}{1-(1-a)G(w(G)-tx)[1+b-bG(w(G)-tx)]}}{1 - \frac{aG(w(G)-t)[1+b-bG(w(G)-t)]}{1-(1-a)G(w(G)-t)[1+b-bG(w(G)-t)]}} \\
= \; & \lim_{t \to 0} \frac{1 - \frac{aG(w(G)-tx)[1+b-b]}{1-(1-a)G(w(G)-tx)[1+b-b]}}{1 - \frac{aG(w(G)-t)}{1-(1-a)G(w(G)-t)[1+b-b]}} \\
= \; & \lim_{t \to 0} \frac{1 - \frac{aG(w(G)-tx)}{1-(1-a)[1+b-b]}}{1 - \frac{aG(w(G)-t)}{1-(1-a)[1+b-b]}} \\
= \; & \lim_{t \to 0} \frac{1 - \frac{aG(w(G)-tx)}{1-(1-a)}}{1 - \frac{aG(w(G)-t)}{1-(1-a)}} \\
= \; & \lim_{t \to 0} \frac{1 - G\left(w(G) - tx\right)}{1 - G(w(G) - t)} \\
= \; & x^{\beta}
\end{aligned}
$$

for every $x > 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left(-(-x)^{\beta}\right)$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

UNSEEN

Solutions to Question B2

ILO: extreme value distribution of a given univariate distribution

a) Note that $w(F) = \infty$ and

$$\begin{aligned} \frac{\Pr(X = k)}{1 - F(k-1)} &= \frac{a(1-a)^{k-1}}{1 - [1 - (1-a)^{k-1}]} \\ &= \frac{a(1-a)^{k-1}}{(1-a)^{k-1}} \\ &= a. \end{aligned}$$

Hence, there can be no non-degenerate limit.

(4 marks)

SEEN

b) Note that $w(F) = 1$ and

$$\begin{aligned} & \lim_{t \to 0} \frac{1 - F(1 - tx)}{1 - F(1-t)} \\ = & \lim_{t \to 0} \frac{x f(1 - tx)}{f(1-t)} \\ = & \lim_{t \to 0} \frac{x \frac{1}{\pi \sqrt{tx(1-tx)}}}{\frac{1}{\pi \sqrt{t(1-t)}}} \\ = & \lim_{t \to 0} x^{1/2} \sqrt{\frac{1-t}{1-tx}} \\ = & x^{1/2}. \end{aligned}$$

So, $F(x)$ belongs to the Weibull domain of attraction.

c) Note that

$$
\begin{aligned}
\lim_{t\downarrow\infty} \frac{1-F(tx)}{1-F(t)} &= \lim_{t\downarrow\infty} \frac{1-\frac{(tx)^b}{a^b+(tx)^b}}{1-\frac{t^b}{a^b+t^b}} \\
&= \lim_{t\downarrow\infty} \frac{\frac{a^b}{a^b+(tx)^b}}{\frac{a^b}{a^b+t^b}} \\
&= \lim_{t\downarrow\infty} \frac{a^b+t^b}{a^b+(tx)^b} \\
&= \lim_{t\downarrow\infty} \frac{a^b t^{-b}+1}{a^b t^{-b}+x^b} \\
&= \frac{0+1}{0+x^b} \\
&= x^{-b}.
\end{aligned}
$$

So, $F(x)$ belongs to the Fréchet domain of attraction.

(4 marks)

UNSEEN

d) Note that

$$
\begin{aligned}
\lim_{t\downarrow\infty} \frac{1-F(tx)}{1-F(t)} &= \lim_{t\downarrow\infty} \frac{xf(tx)}{f(t)} \\
&= \lim_{t\downarrow\infty} \frac{xC(tx-a)^{-\frac{3}{2}} \exp\left[-\frac{1}{2(tx-a)}\right]}{C(t-a)^{-\frac{3}{2}} \exp\left[-\frac{1}{2(t-a)}\right]} \\
&= \lim_{t\downarrow\infty} \frac{x(tx-a)^{-\frac{3}{2}} \exp\left[-\frac{1}{2(tx-a)}\right]}{(t-a)^{-\frac{3}{2}} \exp\left[-\frac{1}{2(t-a)}\right]} \\
&= \lim_{t\downarrow\infty} \frac{x(tx-a)^{-\frac{3}{2}}}{(t-a)^{-\frac{3}{2}}} \\
&= \lim_{t\downarrow\infty} x\left(\frac{tx-a}{t-a}\right)^{-\frac{3}{2}} \\
&= \lim_{t\downarrow\infty} x\left(\frac{x-at^{-1}}{1-at^{-1}}\right)^{-\frac{3}{2}} \\
&= \lim_{t\downarrow\infty} x x^{-\frac{3}{2}} \\
&= \lim_{t\downarrow\infty} x^{-\frac{1}{2}}.
\end{aligned}
$$

So, $F(x)$ belongs to the Fréchet domain of attraction.

(4 marks)

UNSEEN

e) Note that

$$\begin{aligned}
\lim_{t\downarrow\infty}\frac{1-F(tx)}{1-F(t)} &= \lim_{t\downarrow\infty}\frac{xf(tx)}{f(t)} \\
&= \lim_{t\downarrow\infty}\frac{xC\left(1+\frac{t^2x^2}{a}\right)^{-\frac{a+1}{2}}}{C\left(1+\frac{t^2}{a}\right)^{-\frac{a+1}{2}}} \\
&= \lim_{t\downarrow\infty}\frac{x\left(1+\frac{t^2x^2}{a}\right)^{-\frac{a+1}{2}}}{\left(1+\frac{t^2}{a}\right)^{-\frac{a+1}{2}}} \\
&= \lim_{t\downarrow\infty} x\left(\frac{1+\frac{t^2x^2}{a}}{1+\frac{t^2}{a}}\right)^{-\frac{a+1}{2}} \\
&= \lim_{t\downarrow\infty} x\left(\frac{t^{-2}+\frac{x^2}{a}}{t^{-2}+\frac{1}{a}}\right)^{-\frac{a+1}{2}} \\
&= \lim_{t\downarrow\infty} x\left(\frac{t^{-2}+\frac{x^2}{a}}{t^{-2}+\frac{1}{a}}\right)^{-\frac{a+1}{2}} \\
&= xx^{-a-1} \\
&= x^{-a}.
\end{aligned}$$

So, $F(x)$ belongs to the Fréchet domain of attraction.

(4 marks)

UNSEEN

Solutions to Question B3

ILO: estimates of financial risk measures

(a) If X is an absolutely continuous random variable with cumulative distribution function $F(\cdot)$ then

$$\mathrm{VaR}_p(X) = F^{-1}(p)$$

and

$$\mathrm{ES}_p(X) = \frac{1}{p}\int_0^p F^{-1}(v)dv.$$

(2 marks)

SEEN

(b) (i) The corresponding cumulative distribution function is

$$F(x) = \int_{-a}^{x} \frac{3y^2}{2a^3}dy = \frac{3}{2a^3}\left[\frac{y^3}{3}\right]_{-a}^{x} = \frac{3}{2a^3}\left[\frac{x^3+a^3}{3}\right] = \frac{x^3+a^3}{2a^3}$$

for $-a < x < a$;

(2 marks)

UNSEEN

(b) (ii) Inverting

$$F(x) = \frac{x^3+a^3}{2a^3} = p,$$

we obtain $\mathrm{VaR}_p(X) = (2p-1)^{1/3}a$.

(1 marks)

UNSEEN

(b) (iii) The expected shortfall is

$$\mathrm{ES}_p(X) = \frac{1}{p}\int_0^p (2v-1)^{1/3}dv = \frac{1}{p}\left[\frac{3(2v-1)^{4/3}}{8}\right]_0^p = \frac{3a}{8p}\left[(2p-1)^{4/3}-1\right].$$

(2 marks)

UNSEEN

c) (i) The likelihood function of a is

$$\begin{aligned}
L(a) &= \frac{3^n}{2^n a^{3n}} \prod_{i=1}^{n} \left[X_i^2 I\left\{-a < X_i < a\right\} \right] \\
&= \frac{3^n}{2^n a^{3n}} \left(\prod_{i=1}^{n} X_i \right)^2 \prod_{i=1}^{n} I\left\{-a < X_i < a\right\} \\
&= \frac{3^n}{2^n a^{3n}} \left(\prod_{i=1}^{n} X_i \right)^2 I\left\{\max\left(X_1, \ldots, X_n\right) < a, \min\left(X_1, \ldots, X_n\right) > -a\right\} \\
&= \frac{3^n}{2^n a^{3n}} \left(\prod_{i=1}^{n} X_i \right)^2 I\left\{a > \max\left(X_1, \ldots, X_n\right), a > -\min\left(X_1, \ldots, X_n\right)\right\} \\
&= \frac{3^n}{2^n a^{3n}} \left(\prod_{i=1}^{n} X_i \right)^2 I\left\{a > \max\left[\max\left(X_1, \ldots, X_n\right), -\min\left(X_1, \ldots, X_n\right)\right]\right\}.
\end{aligned}$$

(3 marks)

UNSEEN

c) (ii) Note that $L(a)$ is a decreasing function over a and $a > \max\left[\max\left(X_1, \ldots, X_n\right), -\min\left(X_1, \ldots, X_n\right)\right]$. Hence, the mle of a is $\max\left[\max\left(X_1, \ldots, X_n\right), -\min\left(X_1, \ldots, X_n\right)\right]$.

(1 marks)

UNSEEN

c) (iii) The mles of VaR and ES are $\widehat{\mathrm{VaR}}_p(X) = (2p-1)^{1/3}\widehat{a}$ and

$$\widehat{\mathrm{ES}}_p(X) = \frac{3\widehat{a}}{8p}\left[(2p-1)^{4/3} - 1\right],$$

where $\widehat{a} = \max\left[\max\left(X_1, \ldots, X_n\right), -\min\left(X_1, \ldots, X_n\right)\right]$.

(2 marks)

UNSEEN

c (iv) Let $Z = \max\left[\max\left(X_1, \ldots, X_n\right), -\min\left(X_1, \ldots, X_n\right)\right]$. The cumulative distribution function of Z is

$$
\begin{aligned}
F_Z(z) &= \Pr\left(\max\left[\max\left(X_1, \ldots, X_n\right), -\min\left(X_1, \ldots, X_n\right)\right] \leq z\right) \\
&= \Pr\left(\max\left(X_1, \ldots, X_n\right) \leq z, -\min\left(X_1, \ldots, X_n\right) \leq z\right) \\
&= \Pr\left(\max\left(X_1, \ldots, X_n\right) \leq z, \min\left(X_1, \ldots, X_n\right) \geq -z\right) \\
&= \Pr\left(X_1 \leq z, \ldots, X_n \leq z, X_1 \geq -z, \ldots, X_n \geq -z\right) \\
&= \Pr\left(-z \leq X_1 \leq z, \ldots, -z \leq X_n \leq z\right) \\
&= \Pr^n\left(-z \leq X \leq z\right) \\
&= \left[\Pr\left(X \leq z\right) - \Pr\left(X \leq -z\right)\right]^n \\
&= \left[\frac{z^3}{a^3}\right]^n
\end{aligned}
$$

for $z > 0$. The corresponding probability density function is

$$f_Z(z) = 3na^{-3n}z^{3n-1}$$

for $z > 0$.

(3 marks)

UNSEEN

c (v) The expected value of Z is

$$E(Z) = 3na^{-3n}\int_0^a z^{3n}dz = 3na^{-3n}\left[\frac{z^{3n+1}}{3n+1}\right]_0^a dz = 3na^{-3n}\frac{a^{3n+1}}{3n+1} = \frac{3na}{3n+1}.$$

Hence, $\widehat{a}$ is biased for a.

(2 marks)

UNSEEN

c (vi) Since

$$\text{Bias}\left[\widehat{\text{VaR}}_p(X)\right] = (2p-1)^{1/3}\text{Bias}\left[\widehat{a}\right],$$

we see that $\widehat{\text{VaR}}_p(X)$ is biased for $\text{VaR}_p(X)$.

(1 marks)

UNSEEN

c (vii) Since

$$\text{Bias}\left[\widehat{\text{ES}}_p(X)\right] = \frac{3}{8p}\left[(2p-1)^{4/3} - 1\right]\text{Bias}\left[\widehat{a}\right],$$

we see that $\widehat{\mathrm{ES}}_p(X)$ is biased for $\mathrm{ES}_p(X)$.

(1 marks)

UNSEEN

Solutions to Question B4

ILO:probabilities of total portfolio loss and estimates of financial risk measures

(a) (i) T is a $N(\mu_1, \sigma_1^2) + \cdots + N(\mu_k, \sigma_k^2) \equiv N(\mu_1 + \cdots + \mu_k, \sigma_1^2 + \cdots + \sigma_k^2)$ random variable;

(2 marks)

UNSEEN

(a) (ii) Inverting

$$\Phi\left(\frac{t - \mu_1 - \cdots - \mu_k}{\sqrt{\sigma_1^2 + \cdots + \sigma_k^2}}\right) = p,$$

we obtain

$$\mathrm{VaR}_p(T) = \mu_1 + \cdots + \mu_k + \sqrt{\sigma_1^2 + \cdots + \sigma_k^2}\Phi^{-1}(p).$$

(2 marks)

UNSEEN

(a) (iii) The expected shortfall is

$$\begin{aligned}\mathrm{ES}_p(T) &= \frac{1}{p}\int_0^p \left[\mu_1 + \cdots + \mu_k + \sqrt{\sigma_1^2 + \cdots + \sigma_k^2}\Phi^{-1}(v)\right] dv \\ &= \mu_1 + \cdots + \mu_k + \sqrt{\sigma_1^2 + \cdots + \sigma_k^2}\frac{1}{p}\int_0^p \Phi^{-1}(v)dv.\end{aligned}$$

(2 marks)

UNSEEN

b) (i) The joint likelihood function of $\mu_1, \mu_2, \ldots, \mu_k$ and $\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2$ is

$$\begin{aligned}& L\left(\mu_1, \mu_2, \ldots, \mu_k, \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2\right) \\ = & \prod_{i=1}^k \prod_{j=1}^n \left\{\frac{1}{\sqrt{2\pi}\sigma_i} \exp\left[-\frac{(X_{i,j} - \mu_i)^2}{2\sigma_i^2}\right]\right\} \\ = & \prod_{i=1}^k \left\{\frac{1}{(2\pi)^n \sigma_i^n} \exp\left[-\frac{1}{2\sigma_i^2}\sum_{j=1}^n (X_{i,j} - \mu_i)^2\right]\right\} \\ = & \frac{1}{(2\pi)^{nk}\sigma_1^n \cdots \sigma_k^n} \exp\left[-\frac{1}{2}\sum_{i=1}^k \frac{1}{\sigma_i^2}\sum_{j=1}^n (X_{i,j} - \mu_i)^2\right].\end{aligned}$$

(2 marks)

UNSEEN

b) (ii) The log likelihood function is

$$\log L = -nk\log(2\pi) - n\sum_{i=1}^{k}\log\sigma_i - \frac{1}{2}\sum_{i=1}^{k}\frac{1}{\sigma_i^2}\sum_{j=1}^{n}\left(X_{i,j}-\mu_i\right)^2 .$$

The partial derivatives are

$$\frac{\partial \log L}{\partial \mu_i} = \frac{1}{\sigma_i^2}\sum_{j=1}^{n}\left(X_{i,j}-\mu_i\right) = \frac{1}{\sigma_i^2}\left[\left(\sum_{j=1}^{n}X_{i,j}\right) - n\mu_i\right]$$

and

$$\frac{\partial \log L}{\partial \sigma_i} = -\frac{n}{\sigma_i} + \frac{1}{\sigma_i^3}\sum_{j=1}^{n}\left(X_{i,j}-\mu_i\right)^2 .$$

Setting these to zero and solving, we obtain

$$\widehat{\mu_i} = \frac{1}{n}\sum_{j=1}^{n}X_{i,j}$$

and

$$\widehat{\sigma_i^2} = \frac{1}{n}\sum_{j=1}^{n}\left(X_{i,j}-\widehat{\mu_i}\right)^2 .$$

(4 marks)

UNSEEN

b) (iii) $\widehat{\mu_i}$ is unbiased and consistent since

$$E\left(\widehat{\mu_i}\right) = \frac{1}{n}\sum_{j=1}^{n}E\left(X_{i,j}\right) = \frac{1}{n}\sum_{j=1}^{n}\mu_i = \mu_i$$

and

$$Var\left(\widehat{\mu_i}\right) = \frac{1}{n^2}\sum_{j=1}^{n}Var\left(X_{i,j}\right) = \frac{1}{n^2}\sum_{j=1}^{n}\sigma_i^2 = \frac{\sigma_i^2}{n}.$$

(3 marks)

UNSEEN

b) (iv) $\widehat{\sigma_i}^2$ is unbiased and consistent since

$$E\left(\widehat{\sigma_i}^2\right) = \frac{\sigma_i^2}{n} E\left[\frac{1}{\sigma_i^2}\sum_{j=1}^{n}\left(X_{i,j}-\widehat{\mu_i}\right)^2\right] = \frac{\sigma_i^2}{n} E\left[\chi_{n-1}^2\right] = \frac{(n-1)\sigma_i^2}{n}$$

and

$$Var\left(\widehat{\sigma_i}^2\right) = Var\left[\frac{\sigma_i^2}{n}\frac{1}{\sigma_i^2}\sum_{j=1}^{n}\left(X_{i,j}-\widehat{\mu_i}\right)^2\right] = \frac{\sigma_i^4}{n^2} Var\left[\frac{1}{\sigma_i^2}\sum_{j=1}^{n}\left(X_{i,j}-\widehat{\mu_i}\right)^2\right] = \frac{\sigma_i^4}{n^2} Var\left[\chi_{n-1}^2\right] = \frac{2\sigma_i^4(n-1)}{n^2}.$$

(3 marks)

UNSEEN

b (v) The maximum likelihood estimators of $\mathrm{VaR}_p(T)$ and $\mathrm{ES}_p(T)$ are

$$\widehat{\mathrm{VaR}}_p(T) = \frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n} X_{i,j} + \sqrt{\frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n}\left(X_{i,j}-\widehat{\mu_i}\right)^2}\Phi^{-1}(p)$$

and

$$\widehat{\mathrm{ES}}_p(T) = \frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n} X_{i,j} + \sqrt{\frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n}\left(X_{i,j}-\widehat{\mu_i}\right)^2}\frac{1}{p}\int_0^p \Phi^{-1}(v)dv.$$

(2 marks)

UNSEEN

Solutions to Question B5

ILO:probabilities of minimum portfolio loss and estimates of financial risk measures

a) Note that

$$\begin{aligned} F_V(v) &= \Pr(V \leq v) \\ &= 1 - \Pr(V > v) \\ &= 1 - \Pr\left(\min\left(X_1, X_2, \ldots, X_k\right) > v\right) \\ &= 1 - \Pr\left(X_1 > v, X_2 > v, \ldots, X_k > v\right) \\ &= 1 - \overline{F}\left(v, v, \ldots, v\right) \\ &= 1 - \exp\left[-\sum_{i=1}^{k} v - \lambda \max\left(v, v, \ldots, v\right)\right] \\ &= 1 - \exp\left[-kv - \lambda v\right] \end{aligned}$$

for $v > 0$.

(6 marks)

UNSEEN

b) The corresponding probability density function is

$$f_V(v) = (k + \lambda) \exp\left[-kv - \lambda v\right]$$

for $v > 0$.

(2 marks)

UNSEEN

c) Inverting

$$1 - \exp\left[-kv - \lambda v\right] = p$$

gives

$$\mathrm{VaR}_p(V) = -\frac{\log(1-p)}{k + \lambda}.$$

(2 marks)

UNSEEN

d) The expected shortfall is

$$\begin{aligned}
\mathrm{ES}_p(V) &= -\frac{1}{p(k+\lambda)} \int_0^p \log(1-u) du \\
&= -\frac{1}{p(k+\lambda)} \left\{ [u \log(1-u)]_0^p + \int_0^p \frac{u}{1-u} du \right\} \\
&= -\frac{1}{p(k+\lambda)} \left\{ p \log(1-p) + \int_0^p \frac{u-1+1}{1-u} du \right\} \\
&= -\frac{1}{p(k+\lambda)} \left\{ p \log(1-p) + [-u - \log(1-u)]_0^p \right\} \\
&= -\frac{1}{p(k+\lambda)} \left\{ p \log(1-p) - p - \log(1-p) \right\}.
\end{aligned}$$

(2 marks)

UNSEEN

e) The likelihood function is

$$L(\lambda) = \prod_{i=1}^{n} \left\{ (k+\lambda) \exp\left[-k v_i - \lambda v_i\right] \right\} = (k+\lambda)^n \exp\left[-(k+\lambda) \sum_{i=1}^{n} v_i \right].$$

Its log is

$$\log L(\lambda) = n \log(k+\lambda) - (k+\lambda) \sum_{i=1}^{n} v_i.$$

The derivative with respect to λ is

$$\frac{d \log L(\lambda)}{d\lambda} = \frac{n}{k+\lambda} - \sum_{i=1}^{n} v_i.$$

Setting to zero and solving for λ, we obtain

$$\widehat{\lambda} = \frac{n}{\sum_{i=1}^{n} v_i} - k.$$

This is an MLE since

$$\frac{d^2 \log L(\lambda)}{d\lambda^2} = -\frac{n}{(k+\lambda)^2} < 0.$$

(8 marks)

UNSEEN