SOLUTIONS TO MATH68181 EXTREME VALUES AND FINANCIAL RISK EXAM

Solutions to Question A1

ILO: bivariate extreme value distributions

a) The joint cumulative distribution function of X and Y is

$$\begin{aligned} F_{X,Y}(x,y) &= 1 - \overline{F}_{X,Y}(x,0) - \overline{F}_{X,Y}(0,y) + \overline{F}_{X,Y}(x,y) \\ &= 1 - \exp\left[-(a+c)x\right] - \exp\left[-(b+c)y\right] + \exp\left[-ax - by - c\max(x,y)\right]. \end{aligned}$$

(3 marks)

UNSEEN

b) The marginal cumulative distribution functions of X and Y are

$$F_X(x) = F_{X,Y}(x,\infty) = 1 - \exp\left[-(a+c)x\right]$$

and

$$F_Y(y) = F_{X,Y}(\infty,y) = 1 - \exp\left[-(b+c)y\right].$$

(2 marks)

UNSEEN

c) First note that $w\left(F_X\right) = \infty$. F_X belongs to the Gumbel max domain of attraction since

$$\begin{aligned} \lim_{t\to\infty} \frac{1 - F_X\left(t + xh(t)\right)}{1 - F_X(t)} &= \lim_{t\to\infty} \frac{1 - \left[1 - \exp\left(-(a+c)\left(t + xh(t)\right)\right)\right]}{1 - \left[1 - \exp\left(-(a+c)t\right)\right]} \\ &= \lim_{t\to\infty} \frac{\exp\left(-(a+c)\left(t + xh(t)\right)\right)}{\exp\left(-(a+c)t\right)} \\ &= \lim_{t\to\infty} \exp\left(-(a+c)xh(t)\right) \\ &= \exp(-x) \end{aligned}$$

if $h(t) = 1/(a+c)$.

(2 marks)

UNSEEN

d) First note that $w\left(F_Y\right)=\infty$. F_Y belongs to the Gumbel max domain of attraction since

$$\begin{aligned}
\lim_{t\to\infty}\frac{1-F_Y\left(t+yh(t)\right)}{1-F_Y(t)} &= \lim_{t\to\infty}\frac{1-\left[1-\exp\left(-(b+c)\left(t+yh(t)\right)\right)\right]}{1-\left[1-\exp\left(-(b+c)t\right)\right]} \\
&= \lim_{t\to\infty}\frac{\exp\left(-(b+c)\left(t+yh(t)\right)\right)}{\exp\left(-(b+c)t\right)} \\
&= \lim_{t\to\infty}\exp\left(-(b+c)yh(t)\right) \\
&= \exp(-y)
\end{aligned}$$

if $h(t)=1/(b+c)$.

(2 marks)

UNSEEN

e) Use the formulas $a_n=h\left(F_X^{-1}\left(1-\frac{1}{n}\right)\right)$ and $b_n=F_X^{-1}\left(1-\frac{1}{n}\right)$. Inverting

$$F_X(x)=1-\exp\left[-(a+c)x\right]=p$$

gives

$$x=\frac{-\log(1-p)}{a+c}.$$

So,

$$F_X^{-1}\left(1-\frac{1}{n}\right)=\frac{\log n}{a+c}.$$

Hence,

$$a_n=\frac{1}{a+c}, \qquad b_n=\frac{\log n}{a+c}.$$

(2 marks)

UNSEEN

f) Use the formulas $c_n=h\left(F_Y^{-1}\left(1-\frac{1}{n}\right)\right)$ and $d_n=F_Y^{-1}\left(1-\frac{1}{n}\right)$. Inverting

$$F_Y(y)=1-\exp\left[-(b+c)y\right]=p$$

gives

$$y=\frac{-\log(1-p)}{b+c}.$$

So,

$$F_Y^{-1}\left(1 - \frac{1}{n}\right) = \frac{\log n}{b + c}.$$

Hence,

$$c_n = \frac{1}{b + c}, \qquad\qquad d_n = \frac{\log n}{b + c}.$$

(2 marks)

UNSEEN

g) The limiting cumulative distribution function is

$$
\begin{aligned}
& \lim_{n\to\infty} F^n_{X,Y}\left(a_n x + b_n, c_n y + d_n\right) \\
= & \lim_{n\to\infty} \Big\{ 1 - \exp\left[-(a+c)\left(a_n x + b_n\right)\right] - \exp\left[-(b+c)\left(c_n y + d_n\right)\right] \\
& \quad + \exp\left[-a\left(a_n x + b_n\right) - b\left(c_n y + d_n\right) - c\max\left(a_n x + b_n, c_n y + d_n\right)\right] \Big\}^n \\
= & \lim_{n\to\infty} \Big\{ 1 - \frac{\exp(-x)}{n} - \frac{\exp(-y)}{n} \\
& \quad + \exp\left[-a\left(a_n x + b_n\right) - b\left(c_n y + d_n\right) - c\max\left(a_n x + b_n, c_n y + d_n\right)\right] \Big\}^n \\
= & \lim_{n\to\infty} \Big\{ 1 - \frac{\exp(-x)}{n} - \frac{\exp(-y)}{n} \\
& \quad + \exp\left(-\frac{ax}{a+c} - \frac{by}{b+c}\right) n^{-\frac{a}{a+c}-\frac{b}{b+c}} \min\left(\exp\left(-\frac{cx}{a+c}\right) n^{-\frac{c}{a+c}}, \exp\left(-\frac{cy}{b+c}\right) n^{-\frac{c}{b+c}}\right) \Big\}^n \\
= & \lim_{n\to\infty} \Big\{ 1 - \frac{\exp(-x)}{n} - \frac{\exp(-y)}{n} \\
& \quad + \min\left(\exp\left(-x - \frac{by}{b+c}\right) n^{-1-\frac{b}{b+c}}, \exp\left(-\frac{ax}{b+c} - y\right) n^{-1-\frac{a}{a+c}}\right) \Big\}^n \\
= & \lim_{n\to\infty} \Big\{ 1 - \frac{\exp(-x)}{n} - \frac{\exp(-y)}{n} \\
& \quad + \frac{1}{n} \min\left(\exp\left(-x - \frac{by}{b+c}\right) n^{-\frac{b}{b+c}}, \exp\left(-\frac{ax}{b+c} - y\right) n^{-\frac{a}{a+c}}\right) \Big\}^n \\
= & \lim_{n\to\infty} \exp \Big\{ -\exp(-x) - \exp(-y) \\
& \quad + \min\left(\exp\left(-x - \frac{by}{b+c}\right) n^{-\frac{b}{b+c}}, \exp\left(-\frac{ax}{b+c} - y\right) n^{-\frac{a}{a+c}}\right) \Big\} \\
= & \exp\left\{-\exp(-x) - \exp(-y)\right\}.
\end{aligned}
$$

(5 marks)

UNSEEN

h) Yes, the extremes are completely independent since

$$\begin{aligned}
\lim_{n\to\infty} F^n_{X,Y}\left(a_n x + b_n, c_n y + d_n\right) &= \exp\left\{-\exp(-x) - \exp(-y)\right\} \\
&= \exp\left\{-\exp(-x)\right\} \exp\left\{-\exp(-y)\right\} \\
&= \lim_{n\to\infty} F^n_X\left(a_n x + b_n\right) \lim_{n\to\infty} F^n_Y\left(c_n y + d_n\right).
\end{aligned}$$

(2 marks)

UNSEEN

Solutions to Question A2

ILO: checking a function is a copula

$C(u_1, u_2)$ is a valid copula if

$$C(u, 0) = 0,$$

$$C(0, u) = 0,$$

$$C(1, u) = u,$$

$$C(u, 1) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) \geq 0.$$

(4 marks)

SEEN

a) for the copula defined by $C(u_1, u_2) = \exp\left\{-\left[(-\log u_1)^\theta + (-\log u_2)^\theta\right]^{1/\theta}\right\}$, we have

$$C(u, 0) = \exp\left\{-\left[(-\log u)^\theta + (\infty)^\theta\right]^{1/\theta}\right\} = 0,$$

$$C(0, u) = \exp\left\{-\left[(\infty)^\theta + (-\log u)^\theta\right]^{1/\theta}\right\} = 0,$$

$$C(1, u) = \exp\left\{-\left[0 + (-\log u)^\theta\right]^{1/\theta}\right\} = u,$$

$$C(u, 1) = \exp\left\{-\left[(-\log u)^\theta + 0\right]^{1/\theta}\right\} = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = u_1^{-1} (-\log u_1)^{\theta-1} \left[(-\log u_1)^\theta + (-\log u_2)^\theta\right]^{1/\theta-1}$$
$$\cdot \exp\left\{-\left[(-\log u_1)^\theta + (-\log u_2)^\theta\right]^{1/\theta}\right\} \geq 0$$

and

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = u_2^{-1} (-\log u_2)^{\theta - 1} \left[(-\log u_1)^{\theta} + (-\log u_2)^{\theta} \right]^{1/\theta - 1}$$
$$\cdot \exp \left\{ - \left[(-\log u_1)^{\theta} + (-\log u_2)^{\theta} \right]^{1/\theta} \right\} \geq 0.$$

(4 marks)

UNSEEN

b) for the copula defined by $C(u_1, u_2) = \begin{cases} \max(u_1 + u_2 - 1, t), & \text{if } t \leq u_1 \leq 1, t \leq u_2 \leq 1, \\ \min(u_1, u_2), & \text{otherwise,} \end{cases}$ we have

$$C(u, 0) = \min(u, 0) = 0,$$

$$C(0, u) = \min(0, u) = 0,$$

$$C(1, u) = \begin{cases} \max(1 + u - 1, t), & \text{if } t \leq u \leq 1, \\ \min(1, u), & \text{otherwise} \end{cases} = u,$$

$$C(u, 1) = \begin{cases} \max(u + 1 - 1, t), & \text{if } t \leq u \leq 1, \\ \min(u, 1), & \text{otherwise} \end{cases} = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \begin{cases} \frac{\partial}{\partial u_1} \max(u_1 + u_2 - 1, t), & \text{if } t \leq u_1 \leq 1, t \leq u_2 \leq 1, \\ \frac{\partial}{\partial u_1} \min(u_1, u_2), & \text{otherwise} \end{cases} \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \begin{cases} \frac{\partial}{\partial u_2} \max(u_1 + u_2 - 1, t), & \text{if } t \leq u_1 \leq 1, t \leq u_2 \leq 1, \\ \frac{\partial}{\partial u_2} \min(u_1, u_2), & \text{otherwise} \end{cases} \geq 0.$$

(4 marks)

UNSEEN

c) for the copula defined by $\left\{ \left[\left(u_1^{-\theta} - 1 \right)^{\delta} + \left(u_2^{-\theta} - 1 \right)^{\delta} \right]^{1/\delta} + 1 \right\}^{-1/\theta}$, we have

$$C(u, 0) = \left\{ \left[\left(u^{-\theta} - 1 \right)^{\delta} + \left(0^{-\theta} - 1 \right)^{\delta} \right]^{1/\delta} + 1 \right\}^{-1/\theta} = 0,$$

$$C(0,u) = \left\{ \left[\left(0^{-\theta} - 1\right)^{\delta} + \left(u^{-\theta} - 1\right)^{\delta} \right]^{1/\delta} + 1 \right\}^{-1/\theta} = 0,$$

$$C(1,u) = \left\{ \left[(1-1)^{\delta} + \left(u^{-\theta} - 1\right)^{\delta} \right]^{1/\delta} + 1 \right\}^{-1/\theta} = u,$$

$$C(u,1) = \left\{ \left[\left(u^{-\theta} - 1\right)^{\delta} + (1-1)^{\delta} \right]^{1/\delta} + 1 \right\}^{-1/\theta} = u,$$

$$\begin{aligned} \frac{\partial}{\partial u_1} C(u_1, u_2) &= \left\{ \left[\left(u_1^{-\theta} - 1\right)^{\delta} + \left(u_2^{-\theta} - 1\right)^{\delta} \right]^{1/\delta} + 1 \right\}^{-1/\theta - 1} \\ &\quad \cdot \left[\left(u_1^{-\theta} - 1\right)^{\delta} + \left(u_2^{-\theta} - 1\right)^{\delta} \right]^{1/\delta - 1} \left(u_1^{-\theta} - 1\right)^{\delta - 1} u_1^{-\theta - 1} \geq 0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial u_2} C(u_1, u_2) &= \left\{ \left[\left(u_1^{-\theta} - 1\right)^{\delta} + \left(u_2^{-\theta} - 1\right)^{\delta} \right]^{1/\delta} + 1 \right\}^{-1/\theta - 1} \\ &\quad \cdot \left[\left(u_1^{-\theta} - 1\right)^{\delta} + \left(u_2^{-\theta} - 1\right)^{\delta} \right]^{1/\delta - 1} \left(u_2^{-\theta} - 1\right)^{\delta - 1} u_2^{-\theta - 1} \geq 0. \end{aligned}$$

(4 marks)

UNSEEN

d) for the copula defined by $C(u_1, u_2) = w_1 C_1(u_1, u_2) + w_2 C_2(u_1, u_2) + \cdots + w_p C_p(u_1, u_2)$, we have

$$C(u,0) = w_1 C_1(u,0) + w_2 C_2(u,0) + \cdots + w_p C_p(u,0),$$

$$C(0,u) = w_1 C_1(0,u) + w_2 C_2(0,u) + \cdots + w_p C_p(0,u),$$

$$C(1,u) = w_1 C_1(1,u) + w_2 C_2(1,u) + \cdots + w_p C_p(1,u) = (w_1 + w_2 + \cdots + w_p) u = u,$$

$$C(u,1) = w_1 C_1(u,1) + w_2 C_2(u,1) + \cdots + w_p C_p(u,1) = (w_1 + w_2 + \cdots + w_p) u = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \sum_{k=1}^{p} w_k \frac{\partial}{\partial u_1} C_k(u_1, u_2) \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \sum_{k=1}^{p} w_k \frac{\partial}{\partial u_2} C_k(u_1, u_2) \geq 0.$$

(4 marks)

UNSEEN

Solutions to Question A3

ILO: bivariate extreme value distributions

a) We can write

$$\overline{F}(x,y) = \exp\left\{-(x+y)\left[1-(\theta+\phi)\frac{y}{x+y}+\frac{\theta y^2}{(x+y)^2}+\frac{\phi y^3}{(x+y)^3}\right]\right\}$$

for $x>0$, $y>0$, $\theta \geq 0$, $\phi \geq 0$, $\theta+3\phi \geq 0$, $\theta+\phi \leq 1$ and $\theta+2\phi \leq 1$. This is in the form of

$$\overline{F}(x,y) = \exp\left[-(x+y)A\left(\frac{y}{x+y}\right)\right]$$

with $A(w) = 1-(\theta+\phi)w+\theta w^2+\phi w^3$.

We now check the conditions for $A(\cdot)$. Clearly, $A(0)=1$ and $A(1)=1$.

Also $A(t) \geq 0$ since $\theta+\phi \leq 1$ implies $1-(\theta+\phi)w \leq 1$ for all w.

Also $A(w) \leq 1$ since

$$\begin{aligned}
& A(w) \leq 1 \\
\Leftrightarrow\ & 1-(\theta+\phi)w+\theta w^2+\phi w^3 \leq 1 \\
\Leftrightarrow\ & \theta\left(w^2-w\right)+\phi\left(w^3-w\right) \leq 0 \\
\Leftrightarrow\ & \theta w\left(w-1\right)+\phi w\left(w^2-1\right) \leq 0 \\
\Leftrightarrow\ & \left(\theta+\phi+\phi w\right)w\left(w-1\right) \leq 0.
\end{aligned}$$

Note that $\max(w,1-w)=w$ if $w \in [1/2,1]$. So, for $w \in [1/2,1]$,

$$\begin{aligned}
& A(w) \geq \max(w,1-w) \\
\Leftrightarrow\ & A(w) \geq w \\
\Leftrightarrow\ & 1-(\theta+\phi+1)w+\theta w^2+\phi w^3 \geq 0.
\end{aligned}$$

Let $g(w)=1-(\theta+\phi+1)w+\theta w^2+\phi w^3$. Note that $g^{'}(w)=2\theta w+3\phi w^2-\theta-\phi-1 \leq 0$ for all $w \in [1/2,1]$. But $g(1)=0 \geq 0$, so $A(w) \geq \max(w,1-w)$ for all $w \in [1/2,1]$.

Note that $\max(w,1-w)=1-w$ if $w \in [0,1/2]$. So, for $w \in [0,1/2]$,

$$\begin{aligned}
& A(w) \geq \max(w,1-w) \\
\Leftrightarrow\ & A(w) \geq 1-w \\
\Leftrightarrow\ & (1-\theta-\phi)w+\theta w^2+\phi w^3 \geq 0,
\end{aligned}$$

which holds since $\theta+\phi \leq 1$.

$A(\cdot)$ is convex since

$$A^{'}(w) = 2\theta w+3\phi w^2-\theta-\phi$$

and

$$A^{''}(t) = 2\theta + 6\phi w \geq 0$$

for all w since $\theta + 3\phi \geq 0$.

(6 marks)

UNSEEN

b) the joint cdf is

$$F(x, y) = 1 - \exp(-x) - \exp(-y) + \exp\left\{-(x+y)\left[1 - (\theta + \phi)\frac{y}{x+y} + \frac{\theta y^2}{(x+y)^2} + \frac{\phi y^3}{(x+y)^3}\right]\right\}.$$

(2 marks)

UNSEEN

c) the derivative of joint cdf with respect to x is

$$\frac{\partial F(x,y)}{\partial x} = \exp(-x) + \overline{F}(x, y)\left[\frac{\theta y^2}{(x+y)^2} + \frac{2\phi y^3}{(x+y)^3} - 1\right],$$

so the conditional cdf if Y given $X = x$ is

$$F(y|x) = 1 + \exp(x)\overline{F}(x, y)\left[\frac{\theta y^2}{(x+y)^2} + \frac{2\phi y^3}{(x+y)^3} - 1\right].$$

(4 marks)

UNSEEN

d) the derivative of joint cdf with respect to y is

$$\frac{\partial F(x,y)}{\partial y} = \exp(-y) + \overline{F}(x, y)\left[\frac{2\phi y^3}{(x+y)^3} + \frac{(\theta - 3\phi)y^2}{(x+y)^2} - \frac{2\theta y}{x+y} + \theta + \phi - 1\right],$$

so the conditional cdf if X given $Y = y$ is

$$F(x|y) = 1 + \exp(y)\overline{F}(x, y)\left[\frac{2\phi y^3}{(x+y)^3} + \frac{(\theta - 3\phi)y^2}{(x+y)^2} - \frac{2\theta y}{x+y} + \theta + \phi - 1\right].$$

(4 marks)

UNSEEN

e) the derivative of joint cdf with respect to x and y is

$$\begin{aligned}
f(x,y) &= \frac{\partial F(x,y)}{\partial x \partial y} \\
= \quad & \overline{F}(x,y)\left[\frac{\theta y^2}{(x+y)^2}+\frac{2\phi y^3}{(x+y)^3}-1\right]\left[\frac{2\phi y^3}{(x+y)^3}+\frac{(\theta-3\phi)y^2}{(x+y)^2}-\frac{2\theta y}{x+y}+\theta+\phi-1\right] \\
& +\overline{F}(x,y)\left[\frac{2\theta y}{(x+y)^2}+\frac{2(3\phi-\theta)y^2}{(x+y)^3}-\frac{6\theta y^3}{(x+y)^4}\right].
\end{aligned}$$

(4 marks)

UNSEEN

Solutions to Question B1

ILO: extreme value distribution of a given univariate distribution

If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cdf of a normalized version of M_n converges to G, i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n\left(a_n x + b_n\right) \to G(x)$$

as $n \to \infty$ then G must be of the same type as (cdfs G and G^* are of the same type if $G^*(x) = G(ax+b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$\begin{array}{rcl} I & : & \Lambda(x) = \exp\left\{-\exp(-x)\right\}, \qquad x \in \Re; \\ II & : & \Phi_\alpha(x) = \left\{\begin{array}{ll} 0 & \text{if } x < 0, \\ \exp\left\{-x^{-\alpha}\right\} & \text{if } x \geq 0 \end{array}\right. \\ & & \text{for some } \alpha > 0; \\ III & : & \Psi_\alpha(x) = \left\{\begin{array}{ll} \exp\left\{-(-x)^{\alpha}\right\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{array}\right. \\ & & \text{for some } \alpha > 0. \end{array}$$

(4 marks)

SEEN

The necessary and sufficient conditions for the three extreme value distributions are:

$$\begin{array}{rcl} I & : & \exists \gamma(t) > 0 \text{ s.t. } \lim\limits_{t \uparrow w(F)} \dfrac{1 - F\left(t + x\gamma(t)\right)}{1 - F(t)} = \exp(-x), \qquad x > 0, \\ II & : & w(F) = \infty \text{ and } \lim\limits_{t \uparrow \infty} \dfrac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \qquad x > 0, \\ III & : & w(F) < \infty \text{ and } \lim\limits_{t \downarrow 0} \dfrac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} = x^{\alpha}, \qquad x > 0. \end{array}$$

(4 marks)

SEEN

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function say $h(t)$ such that

$$\lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} = e^{-x}$$

for every $x > 0$. But

$$
\begin{aligned}
& \lim_{t \to w(F)} \frac{1 - F\left(t + xh(t)\right)}{1 - F(t)} \\
= & \lim_{t \to w(F)} \frac{\left\{1 - \left[(1+\lambda) G\left(t + xh(t)\right) - \lambda \left(G\left(t + xh(t)\right)\right)^2\right]^a\right\}^b}{\left\{1 - \left[(1+\lambda) G\left(t\right) - \lambda \left(G\left(t\right)\right)^2\right]^a\right\}^b} \\
= & \lim_{t \to w(F)} \left\{\frac{1 - \left[(1+\lambda) G\left(t + xh(t)\right) - \lambda \left(G\left(t + xh(t)\right)\right)^2\right]^a}{1 - \left[(1+\lambda) G\left(t\right) - \lambda \left(G\left(t\right)\right)^2\right]^a}\right\}^b \\
= & \lim_{t \to w(F)} \left\{\frac{1 - \left(G\left(t + xh(t)\right)\right)^a \left[(1+\lambda) - \lambda G\left(t + xh(t)\right)\right]^a}{1 - \left(G\left(t\right)\right)^a \left[(1+\lambda) - \lambda G\left(t\right)\right]^a}\right\}^b \\
= & \lim_{t \to w(G)} \left\{\frac{1 - \left(G\left(t + xh(t)\right)\right)^a \left[(1+\lambda) - \lambda G\left(t + xh(t)\right)\right]^a}{1 - \left(G\left(t\right)\right)^a \left[(1+\lambda) - \lambda G\left(t\right)\right]^a}\right\}^b \\
= & \lim_{t \to w(G)} \left\{\frac{1 - \left[(1+\lambda) - \lambda G\left(t + xh(t)\right)\right]^a}{1 - \left[(1+\lambda) - \lambda G\left(t\right)\right]^a}\right\}^b \\
= & \lim_{t \to w(G)} \left\{\frac{1 - \left[1 + \lambda\left(1 - G\left(t + xh(t)\right)\right)\right]^a}{1 - \left[1 + \lambda\left(1 - G\left(t\right)\right)\right]^a}\right\}^b \\
= & \lim_{t \to w(G)} \left\{\frac{1 - \left[1 + a\lambda\left(1 - G\left(t + xh(t)\right)\right)\right]}{1 - \left[1 + a\lambda\left(1 - G\left(t\right)\right)\right]}\right\}^b \\
= & \lim_{t \to w(G)} \left\{\frac{1 - G\left(t + xh(t)\right)}{1 - G\left(t\right)}\right\}^b \\
= & \left\{\exp(-x)\right\}^b \\
= & \exp(-bx)
\end{aligned}
$$

for every $x > 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left[-\exp(-bx)\right]$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t \to \infty} \frac{1 - G\left(tx\right)}{1 - G(t)} = x^{-\beta}$$

for every $x > 0$. But

$$
\begin{aligned}
& \lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} \\
= & \lim_{t \to \infty} \frac{\left\{1 - \left[(1+\lambda) G(tx) - \lambda (G(tx))^2\right]^a\right\}^b}{\left\{1 - \left[(1+\lambda) G(t) - \lambda (G(t))^2\right]^a\right\}^b} \\
= & \lim_{t \to \infty} \left\{ \frac{1 - \left[(1+\lambda) G(tx) - \lambda (G(tx))^2\right]^a}{1 - \left[(1+\lambda) G(t) - \lambda (G(t))^2\right]^a} \right\}^b \\
= & \lim_{t \to \infty} \left\{ \frac{1 - (G(tx))^a \left[(1+\lambda) - \lambda G(tx)\right]^a}{1 - (G(t))^a \left[(1+\lambda) - \lambda G(t)\right]^a} \right\}^b \\
= & \lim_{t \to \infty} \left\{ \frac{1 - \left[(1+\lambda) - \lambda G(tx)\right]^a}{1 - \left[(1+\lambda) - \lambda G(t)\right]^a} \right\}^b \\
= & \lim_{t \to \infty} \left\{ \frac{1 - \left[1 + \lambda (1 - G(tx))\right]^a}{1 - \left[1 + \lambda (1 - G(t))\right]^a} \right\}^b \\
= & \lim_{t \to \infty} \left\{ \frac{1 - \left[1 + a\lambda (1 - G(tx))\right]}{1 - \left[1 + a\lambda (1 - G(t))\right]} \right\}^b \\
= & \lim_{t \to \infty} \left\{ \frac{1 - G(tx)}{1 - G(t)} \right\}^b \\
= & \left\{ x^{-\beta} \right\}^b \\
= & x^{-b\beta}
\end{aligned}
$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$
\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x \right) = \exp\left(-x^{-b\beta} \right)
$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$
\lim_{t \to 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} = x^{\beta}
$$

for every $x > 0$.

$$
\begin{aligned}
& \lim_{t \to 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} \\
= & \lim_{t \to 0} \frac{\left\{1 - \left[(1+\lambda) G\left(w(F) - tx\right) - \lambda \left(G\left(w(F) - tx\right)\right)^2\right]^a\right\}^b}{\left\{1 - \left[(1+\lambda) G\left(w(F) - t\right) - \lambda \left(G\left(w(F) - t\right)\right)^2\right]^a\right\}^b} \\
= & \lim_{t \to 0} \left\{ \frac{1 - \left[(1+\lambda) G\left(w(F) - tx\right) - \lambda \left(G\left(w(F) - tx\right)\right)^2\right]^a}{1 - \left[(1+\lambda) G\left(w(F) - t\right) - \lambda \left(G\left(w(F) - t\right)\right)^2\right]^a} \right\}^b \\
= & \lim_{t \to 0} \left\{ \frac{1 - \left(G\left(w(F) - tx\right)\right)^a \left[(1+\lambda) - \lambda G\left(w(F) - tx\right)\right]^a}{1 - \left(G\left(w(F) - t\right)\right)^a \left[(1+\lambda) - \lambda G\left(w(F) - t\right)\right]^a} \right\}^b \\
= & \lim_{t \to 0} \left\{ \frac{1 - \left(G\left(w(G) - tx\right)\right)^a \left[(1+\lambda) - \lambda G\left(w(G) - tx\right)\right]^a}{1 - \left(G\left(w(G) - t\right)\right)^a \left[(1+\lambda) - \lambda G\left(w(G) - t\right)\right]^a} \right\}^b \\
= & \lim_{t \to 0} \left\{ \frac{1 - \left[(1+\lambda) - \lambda G\left(w(G) - tx\right)\right]^a}{1 - \left[(1+\lambda) - \lambda G\left(w(G) - t\right)\right]^a} \right\}^b \\
= & \lim_{t \to 0} \left\{ \frac{1 - \left[1 + \lambda\left(1 - G\left(w(G) - tx\right)\right)\right]^a}{1 - \left[1 + \lambda\left(1 - G\left(w(G) - t\right)\right)\right]^a} \right\}^b \\
= & \lim_{t \to 0} \left\{ \frac{1 - \left[1 + a\lambda\left(1 - G\left(w(G) - tx\right)\right)\right]}{1 - \left[1 + a\lambda\left(1 - G\left(w(G) - t\right)\right)\right]} \right\}^b \\
= & \lim_{t \to 0} \left\{ \frac{1 - G\left(w(G) - tx\right)}{1 - G\left(w(G) - t\right)} \right\}^b \\
= & \left\{x^\beta\right\}^b \\
= & x^{b\beta}
\end{aligned}
$$

for every $x > 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left(-(-x)^{b\beta}\right)$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

UNSEEN

Solutions to Question B2

ILO: extreme value distribution of a given univariate distribution

a) Note that $w(F) = \infty$ and

$$\begin{aligned}
\frac{\Pr(X = k)}{1 - F(k-1)} &= \frac{a(1-a)^{k-1}}{1 - \left[1 - (1-a)^{k-1}\right]} \\
&= \frac{a(1-a)^{k-1}}{(1-a)^{k-1}} \\
&= a.
\end{aligned}$$

Hence, there can be no non-degenerate limit.

(4 marks)

SEEN

b) Note that $w(F) = \infty$ and

$$\begin{aligned}
& \lim_{t \to \infty} \frac{1 - F(t + xh(t))}{1 - F(t)} \\
= \ & \lim_{t \to \infty} \frac{1 - \frac{1 - \exp(-t - xh(t))}{1 + \exp(-t - xh(t))}}{1 - \frac{1 - \exp(-t)}{1 + \exp(-t)}} \\
= \ & \lim_{t \to \infty} \frac{1 - \frac{1 - \exp(-t - xh(t))}{1 + 0}}{1 - \frac{1 - \exp(-t)}{1 + 0}} \\
= \ & \lim_{t \to \infty} \frac{\exp(-t - xh(t))}{\exp(-t)} \\
= \ & \lim_{t \to \infty} \exp(-xh(t)) \\
= \ & \exp(-x)
\end{aligned}$$

if $h(t) = 1$. So, $F(x)$ belongs to the Gumbel domain of attraction.

(4 marks)

UNSEEN

c) Note that $w(F) = \infty$ and

$$
\begin{aligned}
& \lim_{t \to \infty} \frac{1 - F(t + xh(t))}{1 - F(t)} \\
= & \lim_{t \to \infty} \frac{\left(1 + xh^{'}(t)\right) f(t + xh(t))}{f(t)} \\
= & \lim_{t \to \infty} \frac{\left(1 + xh^{'}(t)\right) (t + xh(t))^{2a-1} \exp\left[-(t + xh(t))^2\right]}{t^{2a-1} \exp\left[-t^2\right]} \\
= & \lim_{t \to \infty} \left(1 + xh^{'}(t)\right) \left(1 + \frac{xh(t)}{t}\right)^{2a-1} \exp\left[t^2 - (t + xh(t))^2\right] \\
= & \lim_{t \to \infty} \left(1 + xh^{'}(t)\right) \left(1 + \frac{xh(t)}{t}\right)^{2a-1} \exp\left[t^2 - \left(t^2 + x^2 h^2(t) + 2txh(t)\right)\right] \\
= & \lim_{t \to \infty} \left(1 + xh^{'}(t)\right) \left(1 + \frac{xh(t)}{t}\right)^{2a-1} \exp\left[-x^2 h^2(t) - 2txh(t)\right] \\
= & \exp(-x)
\end{aligned}
$$

if $h(t) = \frac{1}{2t}$. So, $F(x)$ belongs to the Gumbel domain of attraction.

d) Note that $w(F) = \infty$. Note that

$$
\begin{aligned}
\lim_{t \downarrow \infty} \frac{1 - F(t + xh(t))}{1 - F(t)} & = \frac{1 - \left[1 - \exp\left(-(t + xh(t))^a\right)\right]^b}{1 - \left[1 - \exp\left(-t^a\right)\right]^b} \\
& = \frac{1 - \left[1 - b\exp\left(-(t + xh(t))^a\right)\right]}{1 - \left[1 - b\exp\left(-t^a\right)\right]} \\
& = \frac{\exp\left(-(t + xh(t))^a\right)}{\exp\left(-t^a\right)} \\
& = \exp\left[t^a - (t + xh(t))^a\right] \\
& = \exp\left\{t^a \left[1 - \left(1 + x\frac{h(t)}{t}\right)^a\right]\right\} \\
& = \exp\left\{t^a \left[1 - \left(1 + ax\frac{h(t)}{t}\right)\right]\right\} \\
& = \exp\left\{-at^{a-1} xh(t)\right\} \\
& = \exp(-x)
\end{aligned}
$$

if $h(t) = \frac{1}{a} t^{1-a}$. So, $F(x)$ belongs to the Gumbel domain of attraction.

(4 marks)

UNSEEN

e) Note that $w(F) = \infty$. Then Note that

$$
\begin{aligned}
\lim_{t \downarrow \infty} \frac{1 - F\left(t + xh(t)\right)}{1 - F(t)} &= \frac{\frac{\log[1-(1-p)\exp(-t-xh(t))]}{\log p}}{\frac{\log[1-(1-p)\exp(-t)]}{\log p}} \\
&= \frac{\log\left[1 - (1-p)\exp\left(-t - xh(t)\right)\right]}{\log\left[1 - (1-p)\exp\left(-t\right)\right]} \\
&= \frac{(1-p)\exp\left(-t - xh(t)\right)}{(1-p)\exp\left(-t\right)} \\
&= \exp\left(-xh(t)\right) \\
&= \exp(-x)
\end{aligned}
$$

if $h(t) = 1$. So, $F(x)$ belongs to the Gumbel domain of attraction.

(4 marks)

UNSEEN

Solutions to Question B3

ILO:probabilities of maximum portfolio loss and estimates of financial risk measures

(a) If X is an absolutely continuous random variable with cdf $F(\cdot)$ then

$$\mathrm{VaR}_p(X) = F^{-1}(p)$$

and

$$\mathrm{ES}_p(X) = \frac{1}{p} \int_0^p F^{-1}(v) dv.$$

(2 marks)

SEEN

(b) (i) The cumulative distribution function of V is

$$\begin{aligned} F_V(v) &= \Pr(V \leq v) \\ &= 1 - \Pr(V > v) \\ &= 1 - \Pr\left(\min(X, Y) > v\right) \\ &= 1 - \Pr\left(X > v, Y > v\right) \\ &= 1 - \overline{F}\left(v, v\right) \\ &= 1 - \exp\left(-v - v - \theta v\right) \\ &= 1 - \exp\left(-2v - \theta v^2\right). \end{aligned}$$

(5 marks)

UNSEEN

(b) (ii) The probability density function of V is

$$f_V(v) = \frac{d}{dv} F_V(v) = 2(1 + \theta v) \exp\left(-2v - \theta v^2\right).$$

(3 marks)

UNSEEN

(b) (iii) Setting

$$1 - \exp\left(-2v - \theta v^2\right) = p$$

implies

$$\exp\left(-2v - \theta v^2\right) = 1 - p$$

which implies

$$2v + \theta v^2 + \log(1-p) = 0$$

which can be solved for v as

$$v = \frac{-2 \pm \sqrt{4 - 4\theta \log(1-p)}}{2\theta} = \frac{-1 \pm \sqrt{1 - \theta \log(1-p)}}{\theta}.$$

Since v must be positive,

$$\mathrm{VaR}_p(V) = \frac{-1 + \sqrt{1 - \theta \log(1-p)}}{\theta}.$$

(5 marks)

UNSEEN

(b) (iv) The likelihood function of θ is

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} \left[2\left(1 + \theta v_i\right) \exp\left(-2v_i - \theta v_i^2\right)\right] \\ &= 2^n \left[\prod_{i=1}^{n} \left(1 + \theta v_i\right)\right] \exp\left(-2\sum_{i=1}^{n} v_i - \theta \sum_{i=1}^{n} v_i^2\right). \end{aligned}$$

Its log is

$$\log L(\theta) = n \log 2 + \sum_{i=1}^{n} \log\left(1 + \theta v_i\right) - 2\sum_{i=1}^{n} v_i - \theta \sum_{i=1}^{n} v_i^2.$$

The derivative with respect to θ is

$$\frac{d \log L(\theta)}{d\theta} = \sum_{i=1}^{n} \frac{v_i}{1 + \theta v_i} - \sum_{i=1}^{n} v_i^2.$$

Setting this to zero gives

$$\sum_{i=1}^{n} \frac{v_i}{1 + \theta v_i} = \sum_{i=1}^{n} v_i^2.$$

The solution of this equation must be an MLE since

$$\frac{d^2 \log L(\theta)}{d\theta^2} = -\sum_{i=1}^{n} \frac{v_i^2}{\left(1 + \theta v_i\right)^2} < 0.$$

(5 marks)

UNSEEN

Solutions to Question B4

ILO:probabilities of minimum portfolio loss and estimates of financial risk measures

a) Note that

$$\begin{aligned}
F_U(u) &= \Pr(U \leq u) \\
&= \Pr\left(\max\left(X_1, X_2, \ldots, X_k\right) \leq u\right) \\
&= \Pr\left(X_1 \leq u, X_2 \leq u, \ldots, X_k \leq u\right) \\
&= F\left(u, u, \ldots, u\right) \\
&= \left[\frac{1}{1 + \displaystyle\sum_{i=1}^{k} \exp\left(-u\right)}\right]^c \\
&= \left[\frac{1}{1 + k\exp\left(-u\right)}\right]^c
\end{aligned}$$

for $-\infty < u < \infty$.

(8 marks)

UNSEEN

b) The corresponding pdf is

$$f_U(u) = ck\exp(-u)\left[1 + k\exp\left(-u\right)\right]^{-c-1}$$

for $-\infty < u < \infty$.

(2 marks)

UNSEEN

c) Inverting

$$\left[\frac{1}{1 + k\exp\left(-u\right)}\right]^c = p$$

gives

$$\mathrm{VaR}_p(U) = -\log\left(p^{-1/c} - 1\right) + \log k.$$

(2 marks)

UNSEEN

d) The likelihood function of c is

$$\begin{aligned} L(c) &= \prod_{i=1}^{n} \left\{ ck \exp\left(-u_i\right) \left[1 + k \exp\left(-u_i\right)\right]^{-c-1} \right\} \\ &= c^n k^n \exp\left(-\sum_{i=1}^{n} u_i\right) \left\{ \prod_{i=1}^{n} \left[1 + k \exp\left(-u_i\right)\right] \right\}^{-c-1}. \end{aligned}$$

Its log is

$$\log L(c) = n \log c + n \log k - \sum_{i=1}^{n} u_i - (c+1) \sum_{i=1}^{n} \log\left[1 + k \exp\left(-u_i\right)\right].$$

The derivative with respect to c is

$$\frac{d \log L(c)}{dc} = \frac{n}{c} - \sum_{i=1}^{n} \log\left[1 + k \exp\left(-u_i\right)\right].$$

Setting this to zero and solving for c gives

$$\widehat{c} = n \left\{ \sum_{i=1}^{n} \log\left[1 + k \exp\left(-u_i\right)\right] \right\}^{-1}.$$

This is an MLE since

$$\frac{d^2 \log L(c)}{dc^2} = -\frac{n}{c^2} < 0.$$

(8 marks)

UNSEEN

Solutions to Question B5

ILO:probabilities of total portfolio loss and estimates of financial risk measures

a) The probability density function of S is

$$
\begin{aligned}
f_S(s) &= \int_0^s f(x, s-x)\,dx \\
&= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \int_0^s x^{a-1}(s-x)^{b-1}(1-s)^{c-1}dx \\
&= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}(1-s)^{c-1}s^{a+b-1} \int_0^s \left(\frac{x}{s}\right)^{a-1}\left(1-\frac{x}{s}\right)^{b-1} d(x/s) \\
&= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}(1-s)^{c-1}s^{a+b-1} \int_0^1 u^{a-1}(1-u)^{b-1}\,du \\
&= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}(1-s)^{c-1}s^{a+b-1}B(a,b) \\
&= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}(1-s)^{c-1}s^{a+b-1}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \\
&= \frac{\Gamma(a+b+c)}{\Gamma(a+b)\Gamma(c)}s^{a+b-1}(1-s)^{c-1}
\end{aligned}
$$

for $0 < s < 1$.

(8 marks)

UNSEEN

b) The corresponding cumulative distribution function is

$$
\begin{aligned}
F_S(s) &= \frac{\Gamma(a+b+c)}{\Gamma(a+b)\Gamma(c)} \int_0^s t^{a+b-1}(1-t)^{c-1}dt \\
&= \frac{1}{B(a+b,c)}B_s(a+b,c) \\
&= I_s(a+b,c)
\end{aligned}
$$

for $0 < s < 1$.

(2 marks)

UNSEEN

c) Inverting

$$I_s(a+b,c) = p$$

gives

$$\mathrm{VaR}_p(S) = I_p^{-1}(a+b, c).$$

(2 marks)

UNSEEN

e) The likelihood function is

$$\begin{aligned} L\left(a, b, c\right) &= \prod_{i=1}^{n} \left\{ \frac{\Gamma(a+b+c)}{\Gamma(a+b)\Gamma(c)} s_i^{a+b-1} \left(1 - s_i\right)^{c-1} \right\} \\ &= \frac{\Gamma^n(a+b+c)}{\Gamma^n(a+b)\Gamma^n(c)} \left(\prod_{i=1}^{n} s_i \right)^{a+b-1} \left[\prod_{i=1}^{n} \left(1 - s_i\right) \right]^{c-1}. \end{aligned}$$

Its log is

$$\log L\left(a, b, c\right) = n \log \Gamma(a+b+c) - n \log \Gamma(a+b) - n \log \Gamma(c) + (a+b-1) \sum_{i=1}^{n} \log s_i + (c-1) \sum_{i=1}^{n} \log (1 -$$

The derivatives with respect to a, b and c are

$$\frac{d \log L\left(\lambda\right)}{da} = n \frac{d}{da} \log \Gamma(a+b+c) - n \frac{d}{da} \log \Gamma(a+b) + \sum_{i=1}^{n} \log s_i,$$

$$\frac{d \log L\left(\lambda\right)}{db} = n \frac{d}{db} \log \Gamma(a+b+c) - n \frac{d}{db} \log \Gamma(a+b) + \sum_{i=1}^{n} \log s_i$$

and

$$\frac{d \log L\left(\lambda\right)}{dc} = n \frac{d}{dc} \log \Gamma(a+b+c) - n \frac{d}{dc} \log \Gamma(c) + \sum_{i=1}^{n} \log \left(1 - s_i\right).$$

Hence, the MLEs satisfy the equations

$$n \frac{d}{d(a+b+c)} \log \Gamma(a+b+c) - n \frac{d}{d(a+b)} \log \Gamma(a+b) + \sum_{i=1}^{n} \log s_i = 0$$

and

$$n \frac{d}{d(a+b+c)} \log \Gamma(a+b+c) - n \frac{d}{dc} \log \Gamma(c) + \sum_{i=1}^{n} \log \left(1 - s_i\right) = 0.$$

(8 marks)

UNSEEN