SOLUTIONS TO MATH68181 EXTREME VALUES AND FINANCIAL RISK EXAM

Solutions to Question A1

a) The joint cdf of X and Y is

$$\begin{aligned} F_{X,Y}(x,y) &= 1-\overline{F}_{X,Y}(x,0)-\overline{F}_{X,Y}(0,y)+\overline{F}_{X,Y}(x,y) \\ &= 1-(1+ax)^{-q}-(1+by)^{-q}+(1+ax+by+cxy)^{-q}. \end{aligned}$$

(3 marks)

UNSEEN

b) The marginal cdfs of X and Y are

$$F_X(x)=F_{X,Y}(x,\infty)=1-(1+ax)^{-q}$$

and

$$F_Y(y)=F_{X,Y}(\infty,y)=1-(1+by)^{-q}.$$

(2 marks)

UNSEEN

c) First note that $w(F_X)=\infty$. F_X belongs to the Fréchet max domain of attraction since

$$\begin{aligned} \lim_{t\to\infty}\frac{1-F_X(tx)}{1-F_X(t)} &= \lim_{t\to\infty}\frac{1-\left[1-(1+atx)^{-q}\right]}{1-\left[1-(1+at)^{-q}\right]} \\ &= \lim_{t\to\infty}\frac{(1+atx)^{-q}}{(1+at)^{-q}} \\ &= \lim_{t\to\infty}\left(\frac{1+atx}{1+at}\right)^{-q} \\ &= \lim_{t\to\infty}\left(\frac{\frac{1}{t}+ax}{\frac{1}{t}+a}\right)^{-q} \\ &= \left(\frac{0+ax}{0+a}\right)^{-q} \\ &= x^{-q}. \end{aligned}$$

(2 marks)

UNSEEN

d) First note that $w\left(F_Y\right)=\infty$. F_Y belongs to the Fréchet max domain of attraction since

$$\begin{aligned}
\lim_{t\to\infty}\frac{1-F_Y\left(ty\right)}{1-F_Y(t)} &= \lim_{t\to\infty}\frac{1-\left[1-(1+bty)^{-q}\right]}{1-\left[1-(1+bt)^{-q}\right]}\\
&= \lim_{t\to\infty}\frac{(1+bty)^{-q}}{(1+bt)^{-q}}\\
&= \lim_{t\to\infty}\left(\frac{1+bty}{1+bt}\right)^{-q}\\
&= \lim_{t\to\infty}\left(\frac{\frac{1}{t}+by}{\frac{1}{t}+b}\right)^{-q}\\
&= \left(\frac{0+by}{0+b}\right)^{-q}\\
&= y^{-q}.
\end{aligned}$$

(2 marks)

UNSEEN

e) Use the formulas $a_n=F_X^{-1}\left(1-\frac{1}{n}\right)$ and $b_n=0$. Inverting

$$F_X(x)=1-(1+ax)^{-q}=p$$

gives

$$x=\frac{(1-p)^{-1/q}-1}{a}.$$

So,

$$F_X^{-1}\left(1-\frac{1}{n}\right)=\frac{n^{1/q}-1}{a}.$$

Hence,

$$a_n=\frac{n^{1/q}-1}{a}$$

and $b_n=0$.

(2 marks)

UNSEEN

f) Use the formulas $c_n=F_Y^{-1}\left(1-\frac{1}{n}\right)$ and $d_n=0$. Inverting

$$F_Y(y)=1-(1+by)^{-q}=p$$

gives

$$y = \frac{(1-p)^{-1/q} - 1}{b}.$$

So,

$$F_Y^{-1}\left(1 - \frac{1}{n}\right) = \frac{n^{1/q} - 1}{b}.$$

Hence,

$$c_n = \frac{n^{1/q} - 1}{b}$$

and $b_n = 0$.

(2 marks)

UNSEEN

g) The limiting cdf is

$$\begin{aligned}
& \lim_{n\to\infty} F_{X,Y}^n \left(a_n x + b_n, c_n y + d_n\right) \\
= & \lim_{n\to\infty} \left[1 - (1 + a_n a x)^{-q} - (1 + c_n b y)^{-q} + (1 + a_n a x + c_n b y + c a_n c_n x y)^{-q}\right]^n \\
= & \lim_{n\to\infty} \left\{1 - \left[1 + \left(n^{1/q} - 1\right) x\right]^{-q} - \left[1 + \left(n^{1/q} - 1\right) y\right]^{-q} \right. \\
& \qquad \left. + \left[1 + \left(n^{1/q} - 1\right) x + \left(n^{1/q} - 1\right) y + \frac{c}{ab}\left(n^{1/q} - 1\right)^2 xy\right]^{-q}\right\}^n \\
= & \lim_{n\to\infty} \left\{1 - \frac{1}{n}\left[x + (1-x) n^{-1/q}\right]^{-q} - \frac{1}{n}\left[y + (1-y) n^{-1/q}\right]^{-q} \right. \\
& \qquad \left. + \frac{1}{n^2}\left[n^{-2/q} + \left(n^{-1/q} - n^{-2/q}\right) x + \left(n^{-1/q} - n^{-2/q}\right) y + \frac{c}{ab}\left(1 - n^{-1/q}\right)^2 xy\right]^{-q}\right\}^n \\
= & \lim_{n\to\infty} \left\{1 - \frac{1}{n} x^{-q} - \frac{1}{n} y^{-q} + \frac{1}{n^2}\left[\frac{c}{ab} xy\right]^{-q}\right\}^n \\
= & \lim_{n\to\infty} \left\{1 - \frac{1}{n} x^{-q} - \frac{1}{n} y^{-q}\right\}^n \\
= & \exp\left(-x^{-q} - y^{-q}\right).
\end{aligned}$$

(5 marks)

UNSEEN

h) Yes, the extremes are completely independent since

$$
\begin{aligned}
\lim_{n\to\infty} F^n_{X,Y}\left(a_n x + b_n, c_n y + d_n\right) &= \exp\left(-x^{-q} - y^{-q}\right) \\
&= \exp\left(-x^{-q}\right)\exp\left(-y^{-q}\right). \\
&= \lim_{n\to\infty} F^n_X\left(a_n x + b_n\right) \lim_{n\to\infty} F^n_Y\left(c_n y + d_n\right).
\end{aligned}
$$

(2 marks)

UNSEEN

Solutions to Question A2

$C(u_1, u_2)$ is a valid copula if

$$C(u, 0) = 0,$$

$$C(0, u) = 0,$$

$$C(1, u) = u,$$

$$C(u, 1) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) \geq 0.$$

(4 marks)

SEEN

a) for the copula defined by $C(u_1, u_2) = u_1^{1-a} u_2^{1-b} \min\left(u_1^a, u_2^b\right)$, we have

$$C(u, 0) = u^{1-a} 0^{1-b} \min\left(u^a, 0^b\right) = 0,$$

$$C(0, u) = 0^{1-a} u^{1-b} \min\left(0^a, u^b\right) = 0,$$

$$C(1, u) = 1^{1-a} u^{1-b} \min\left(1^a, u^b\right) = u,$$

$$C(u, 1) = u^{1-a} 1^{1-b} \min\left(u^a, 1^b\right) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \frac{\partial}{\partial u_1} \left\{ \begin{array}{ll} u_1 u_2^{1-b}, & \text{if } u_1^a \leq u_2^b, \\ u_1^{1-a} u_2, & \text{if } u_1^a > u_2^b \end{array} \right. = \left\{ \begin{array}{ll} u_2^{1-b}, & \text{if } u_1^a \leq u_2^b, \\ (1-a) u_1^{-a} u_2, & \text{if } u_1^a > u_2^b \end{array} \right. \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \frac{\partial}{\partial u_2} \left\{ \begin{array}{ll} u_1 u_2^{1-b}, & \text{if } u_1^a \leq u_2^b, \\ u_1^{1-a} u_2, & \text{if } u_1^a > u_2^b \end{array} \right. = \left\{ \begin{array}{ll} (1-b) u_1 u_2^{-b}, & \text{if } u_1^a \leq u_2^b, \\ u_1^{1-a}, & \text{if } u_1^a > u_2^b \end{array} \right. \geq 0.$$

(4 marks)

UNSEEN

b) for the copula defined by $C(u_1, u_2) = \frac{u_1 u_2}{u_1+u_2-u_1u_2}$, we have

$$C(u, 0) = \frac{u \cdot 0}{u + 0 - u \cdot 0} = 0,$$

$$C(0, u) = \frac{0 \cdot u}{0 + u - 0 \cdot u} = 0,$$

$$C(1, u) = \frac{1 \cdot u}{1 + u - 1 \cdot u} = u,$$

$$C(u, 1) = \frac{u \cdot 1}{u + 1 - u \cdot 1} = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \frac{u_2^2}{(u_1 + u_2 - u_1 u_2)^2} \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \frac{u_1^2}{(u_1 + u_2 - u_1 u_2)^2} \geq 0.$$

(4 marks)

UNSEEN

c) for the copula defined by $C(u_1, u_2) = [C_1(u_1, u_2) C_2(u_1, u_2) \cdots C_p(u_1, u_2)]^{1/p}$, we have

$$C(u, 0) = [C_1(u, 0) C_2(u, 0) \cdots C_p(u, 0)]^{1/p} = 0,$$

$$C(0, u) = [C_1(0, u) C_2(0, u) \cdots C_p(0, u)]^{1/p} = 0,$$

$$C(1, u) = [C_1(1, u) C_2(1, u) \cdots C_p(1, u)]^{1/p} = [u^p]^{1/p} = u,$$

$$C(u, 1) = [C_1(u, 1) C_2(u, 1) \cdots C_p(u, 1)]^{1/p} = [u^p]^{1/p} = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \frac{1}{p} [C_1(u_1, u_2) C_2(u_1, u_2) \cdots C_p(u_1, u_2)]^{\frac{1}{p}-1} \sum_{k=1}^{p} \left[\prod_{j=1, j \neq k}^{p} C_j(u_1, u_2) \right] \frac{\partial}{\partial u_1} C_k(u_1, u_2) \geq 0$$

and

$$\frac{\partial}{\partial u_2} C\left(u_1, u_2\right) = \frac{1}{p}\left[C_1\left(u_1, u_2\right) C_2\left(u_1, u_2\right) \cdots C_p\left(u_1, u_2\right)\right]^{\frac{1}{p}-1} \sum_{k=1}^{p}\left[\prod_{j=1, j \neq k}^{p} C_j\left(u_1, u_2\right)\right] \frac{\partial}{\partial u_2} C_k\left(u_1, u_2\right) \geq 0.$$

(4 marks)

UNSEEN

d) for the copula defined by $C\left(u_1, u_2\right) = w_1 C_1\left(u_1, u_2\right) + w_2 C_2\left(u_1, u_2\right) + \cdots + w_p C_p\left(u_1, u_2\right)$, we have

$$C(u, 0) = w_1 C_1(u, 0) + w_2 C_2(u, 0) + \cdots + w_p C_p(u, 0),$$

$$C(0, u) = w_1 C_1(0, u) + w_2 C_2(0, u) + \cdots + w_p C_p(0, u),$$

$$C(1, u) = w_1 C_1(1, u) + w_2 C_2(1, u) + \cdots + w_p C_p(1, u) = \left(w_1 + w_2 + \cdots + w_p\right) u = u,$$

$$C(u, 1) = w_1 C_1(u, 1) + w_2 C_2(u, 1) + \cdots + w_p C_p(u, 1) = \left(w_1 + w_2 + \cdots + w_p\right) u = u,$$

$$\frac{\partial}{\partial u_1} C\left(u_1, u_2\right) = \sum_{k=1}^{p} w_k \frac{\partial}{\partial u_1} C_k\left(u_1, u_2\right) \geq 0$$

and

$$\frac{\partial}{\partial u_2} C\left(u_1, u_2\right) = \sum_{k=1}^{p} w_k \frac{\partial}{\partial u_2} C_k\left(u_1, u_2\right) \geq 0.$$

(4 marks)

UNSEEN

Solutions to Question A3

a) We can write

$$\overline{G}(x, y) = \exp\left[-(x+y)A\left(\frac{y}{x+y}\right)\right]$$

for $x > 0$ and $y > 0$, where

$$A(w) = 1 - \alpha w(1-w)\left[1 - \beta w(1-w)\right].$$

We now check the conditions for $A(\cdot)$. Clearly,

$$A(0) = 1$$

and

$$A(1) = 1.$$

Also $A(w) \leq 1$ since $\alpha w(1-w)\left[1 - \beta w(1-w)\right] \geq 0$ which holds since $w(1-w) \leq 1/4$ for all $0 \leq w \leq 1$.

Also

$$\begin{aligned}
& A(w) \geq w \\
& \Longleftrightarrow 1 - \alpha w(1-w)\left[1 - \beta w(1-w)\right] \geq w \\
& \Longleftrightarrow 1 - w - \alpha w(1-w)\left[1 - \beta w(1-w)\right] \geq 0 \\
& \Longleftrightarrow 1 - \alpha w\left[1 - \beta w(1-w)\right] \geq 0 \\
& \Longleftrightarrow 1 - \alpha w + \alpha\beta w^2 - \alpha\beta w^3 = g(w) \text{ say } \geq 0.
\end{aligned}$$

Note that $g^{'}(w) = -\alpha + 2\alpha\beta w - 3\alpha\beta w^2$ and $g^{''}(w) = 2\alpha\beta - 6\alpha\beta w = 2\alpha\beta(1-3w)$. Since $g^{'}(0) = -\alpha$ and $g^{'}(1) = -\alpha - \alpha\beta$, $g(w)$ is a decreasing function for $0 \leq w \leq 1$. Because $g(1) = 1 - \alpha \geq 0$, $g(w) \geq 0$ for all $0 \leq w \leq 1$.

Also

$$\begin{aligned}
& A(w) \geq 1 - w \\
& \Longleftrightarrow 1 - \alpha w(1-w)\left[1 - \beta w(1-w)\right] \geq 1 - w \\
& \Longleftrightarrow w - \alpha w(1-w)\left[1 - \beta w(1-w)\right] \geq 0 \\
& \Longleftrightarrow 1 - \alpha(1-w)\left[1 - \beta w(1-w)\right] \geq 0 \\
& \Longleftrightarrow 1 - \alpha(1-w) + \alpha\beta(1-w)^2 - \alpha\beta(1-w)^3 = g(w) \text{ say } \geq 0.
\end{aligned}$$

Note that $g^{'}(w) = \alpha - 2\alpha\beta(1-w) + 3\alpha\beta(1-w)^2$ and $g^{''}(w) = 2\alpha\beta - 6\alpha\beta(1-w) = 2\alpha\beta(3w-1)$. Since $g^{'}(0) = \alpha + \alpha\beta$ and $g^{'}(1) = \alpha$, $g(w)$ is an increasing function for $0 \leq w \leq 1$. Because $g(10) = 1 - \alpha \geq 0$, $g(w) \geq 0$ for all $0 \leq w \leq 1$.

The first and second derivatives of $A(w)$ are

$$A^{'}(w) = -\alpha + 2\alpha w + 2\alpha\beta w - 6\alpha\beta w^2 + 4\alpha\beta w^3$$

and

$$A^{''}(w) = 2\alpha + 2\alpha\beta - 12\alpha\beta w + 12\alpha\beta w^2,$$

respectively. Note that

$$A^{'''}(w) = -12\alpha\beta + 24\alpha\beta w = 12\alpha\beta(2w - 1).$$

Since

$$A^{''}(1/2) = 2\alpha + 2\alpha\beta - 6\alpha\beta w + 3\alpha\beta = \alpha(2 - \beta) \geq 0,$$

we have $A^{''}(w) \geq 0$ for all $0 \leq w \leq 1$. Hence, $A(w)$ is convex.

(7 marks)

UNSEEN

b) Note that

$$\overline{G}(x, 0) = \exp\left\{-(x+0)A\left(0\right)\right\} = \exp(-x)$$

and

$$\overline{G}(0, y) = \exp\left\{-(0+y)A\left(1\right)\right\} = \exp(-y).$$

So, the joint cdf is

$$G(x, y) = 1 - \exp(-x) - \exp(-y) + \exp\left\{-x - y + \frac{\alpha xy}{x+y}\left[1 - \frac{\beta xy}{(x+y)^2}\right]\right\}.$$

(1 marks)

UNSEEN

c) the derivative of joint cdf with respect to x is

$$\frac{\partial G(x, y)}{\partial x} = \exp(-x) + \overline{G}(x, y)\left[-1 + \frac{\alpha y^2}{(x+y)^2} - \frac{2\alpha\beta xy^2}{(x+y)^3} + \frac{3\alpha\beta x^2y^2}{(x+y)^4}\right],$$

so the conditional cdf of Y given $X = x$ is

$$G(y \mid x) = 1 + \exp\left\{-y + \frac{\alpha xy}{x+y}\left[1 - \frac{\beta xy}{(x+y)^2}\right]\right\}\left[-1 + \frac{\alpha y^2}{(x+y)^2} - \frac{2\alpha\beta xy^2}{(x+y)^3} + \frac{3\alpha\beta x^2y^2}{(x+y)^4}\right].$$

(4 marks)

UNSEEN

d) the derivative of joint cdf with respect to y is

$$\frac{\partial G(x,y)}{\partial y} = \exp(-x) + \overline{G}(x,y)\left[-1 + \frac{\alpha x^2}{(x+y)^2} - \frac{2\alpha\beta y x^2}{(x+y)^3} + \frac{3\alpha\beta x^2 y^2}{(x+y)^4}\right],$$

so the conditional cdf of Y given $Y = y$ is

$$G(y \mid x) = 1 + \exp\left\{-x + \frac{\alpha xy}{x+y}\left[1 - \frac{\beta xy}{(x+y)^2}\right]\right\}\left[-1 + \frac{\alpha x^2}{(x+y)^2} - \frac{2\alpha\beta y x^2}{(x+y)^3} + \frac{3\alpha\beta x^2 y^2}{(x+y)^4}\right].$$

(4 marks)

UNSEEN

e) the derivative of joint cdf with respect to x and y is

$$\begin{aligned} g(x,y) = {} & \overline{G}(x,y)\left[-1 + \frac{\alpha y^2}{(x+y)^2} - \frac{2\alpha\beta x y^2}{(x+y)^3} + \frac{3\alpha\beta x^2 y^2}{(x+y)^4}\right]\left[-1 + \frac{\alpha x^2}{(x+y)^2} - \frac{2\alpha\beta y x^2}{(x+y)^3} + \frac{3\alpha\beta x^2 y^2}{(x+y)^4}\right] \\ & + \overline{G}(x,y)\left[\frac{2\alpha y}{(x+y)^2} - \frac{2\alpha y^2}{(x+y)^3} - \frac{4\alpha\beta xy}{(x+y)^3} + \frac{6\alpha\beta x y^2}{(x+y)^3} + \frac{6\alpha\beta x^2 y}{(x+y)^4} - \frac{12\alpha\beta x^2 y^2}{(x+y)^5}\right]. \end{aligned}$$

(4 marks)

UNSEEN

Solutions to Question B1

If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cdf of a normalized version of M_n converges to G, i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n\left(a_n x + b_n\right) \to G(x)$$

as $n \to \infty$ then G must be of the same type as (cdfs G and G^* are of the same type if $G^*(x) = G(ax+b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$\begin{array}{rcl} I & : & \Lambda(x) = \exp\left\{-\exp(-x)\right\}, \qquad x \in \Re; \\ II & : & \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\left\{-x^{-\alpha}\right\} & \text{if } x \geq 0 \end{cases} \\ & & \text{for some } \alpha > 0; \\ III & : & \Psi_\alpha(x) = \begin{cases} \exp\left\{-(-x)^{\alpha}\right\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \\ & & \text{for some } \alpha > 0. \end{array}$$

(4 marks)

SEEN

The necessary and sufficient conditions for the three extreme value distributions are:

$$\begin{array}{rcl} I & : & \exists \gamma(t) > 0 \text{ s.t. } \lim\limits_{t \uparrow w(F)} \dfrac{1 - F\left(t + x\gamma(t)\right)}{1 - F(t)} = \exp(-x), \qquad x \in \Re, \\ II & : & w(F) = \infty \text{ and } \lim\limits_{t \uparrow \infty} \dfrac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \qquad x > 0, \\ III & : & w(F) < \infty \text{ and } \lim\limits_{t \downarrow 0} \dfrac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} = x^{\alpha}, \qquad x > 0. \end{array}$$

(4 marks)

SEEN

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function say $h(t)$ such that

$$\lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} = e^{-x}$$

for every $x > 0$. But

$$\begin{aligned}
& \lim_{t \to w(F)} \frac{1 - F\left(t + xh(t)\right)}{1 - F(t)} \\
= & \lim_{t \to w(F)} \frac{1 - \frac{[G(t+xh(t))]^a}{[G(t+xh(t))]^a + [1-G(t+xh(t))]^a}}{1 - \frac{[G(t)]^a}{[G(t)]^a + [1-G(t)]^a}} \\
= & \lim_{t \to w(F)} \frac{\frac{[1-G(t+xh(t))]^a}{[G(t+xh(t))]^a + [1-G(t+xh(t))]^a}}{\frac{[1-G(t)]^a}{[G(t)]^a + [1-G(t)]^a}} \\
= & \lim_{t \to w(F)} \frac{\left[1 - G\left(t + xh(t)\right)\right]^a}{[1 - G(t)]^a} \frac{[G(t)]^a + [1 - G(t)]^a}{\left[G\left(t + xh(t)\right)\right]^a + \left[1 - G\left(t + xh(t)\right)\right]^a} \\
= & \lim_{t \to w(G)} \frac{\left[1 - G\left(t + xh(t)\right)\right]^a}{[1 - G(t)]^a} \lim_{t \to w(G)} \frac{[G(t)]^a + [1 - G(t)]^a}{\left[G\left(t + xh(t)\right)\right]^a + \left[1 - G\left(t + xh(t)\right)\right]^a} \\
= & \lim_{t \to w(G)} \frac{\left[1 - G\left(t + xh(t)\right)\right]^a}{[1 - G(t)]^a} \frac{1 + 0}{1 + 0} \\
= & \lim_{t \to w(G)} \frac{\left[1 - G\left(t + xh(t)\right)\right]^a}{[1 - G(t)]^a} \\
= & \lim_{t \to w(G)} \left[\frac{1 - G\left(t + xh(t)\right)}{1 - G(t)}\right]^a \\
= & \exp(-ax)
\end{aligned}$$

for every $x > 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left[-\exp(-ax)\right]$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t \to \infty} \frac{1 - G\left(tx\right)}{1 - G(t)} = x^{-\beta}$$

for every $x > 0$. But

$$
\begin{aligned}
& \lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} \\
= & \lim_{t \to \infty} \frac{1 - \frac{[G(tx)]^a}{[G(tx)]^a + [1 - G(tx)]^a}}{1 - \frac{[G(t)]^a}{[G(t)]^a + [1 - G(t)]^a}} \\
= & \lim_{t \to \infty} \frac{\frac{[1 - G(tx)]^a}{[G(tx)]^a + [1 - G(tx)]^a}}{\frac{[1 - G(t)]^a}{[G(t)]^a + [1 - G(t)]^a}} \\
= & \lim_{t \to \infty} \frac{[1 - G(tx)]^a}{[1 - G(t)]^a} \frac{[G(t)]^a + [1 - G(t)]^a}{[G(tx)]^a + [1 - G(tx)]^a} \\
= & \lim_{t \to \infty} \frac{[1 - G(tx)]^a}{[1 - G(t)]^a} \lim_{t \to \infty} \frac{[G(t)]^a + [1 - G(t)]^a}{[G(tx)]^a + [1 - G(tx)]^a} \\
= & \lim_{t \to \infty} \frac{[1 - G(tx)]^a}{[1 - G(t)]^a} \frac{1 + 0}{1 + 0} \\
= & \lim_{t \to \infty} \frac{[1 - G(tx)]^a}{[1 - G(t)]^a} \\
= & \lim_{t \to \infty} \left[\frac{1 - G(tx)}{1 - G(t)} \right]^a \\
= & x^{-a\beta}
\end{aligned}
$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$
\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x \right) = \exp\left(-x^{-a\beta} \right)
$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$
\lim_{t \to 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} = x^{\beta}
$$

for every $x > 0$.

$$\begin{aligned}
& \lim_{t \to 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} \\
= & \lim_{t \to 0} \frac{1 - \frac{[G(w(F)-tx)]^a}{[G(w(F)-tx)]^a + [1-G(w(F)-tx)]^a}}{1 - \frac{[G(w(F)-t)]^a}{[G(w(F)-t)]^a + [1-G(w(F)-t)]^a}} \\
= & \lim_{t \to 0} \frac{\frac{[1-G(w(F)-tx)]^a}{[G(w(F)-tx)]^a + [1-G(w(F)-tx)]^a}}{\frac{[1-G(w(F)-t)]^a}{[G(w(F)-t)]^a + [1-G(w(F)-t)]^a}} \\
= & \lim_{t \to 0} \frac{\left[1 - G\left(w(F) - tx\right)\right]^a}{\left[1 - G\left(w(F) - t\right)\right]^a} \frac{\left[G\left(w(F) - t\right)\right]^a + \left[1 - G\left(w(F) - t\right)\right]^a}{\left[G\left(w(F) - tx\right)\right]^a + \left[1 - G\left(w(F) - tx\right)\right]^a} \\
= & \lim_{t \to 0} \frac{\left[1 - G\left(w(G) - tx\right)\right]^a}{\left[1 - G\left(w(G) - t\right)\right]^a} \lim_{t \to 0} \frac{\left[G\left(w(G) - t\right)\right]^a + \left[1 - G\left(w(G) - t\right)\right]^a}{\left[G\left(w(G) - tx\right)\right]^a + \left[1 - G\left(w(G) - tx\right)\right]^a} \\
= & \lim_{t \to 0} \frac{\left[1 - G\left(w(G) - tx\right)\right]^a}{\left[1 - G\left(w(G) - t\right)\right]^a} \frac{1 + 0}{1 + 0} \\
= & \lim_{t \to 0} \frac{\left[1 - G\left(w(G) - tx\right)\right]^a}{\left[1 - G\left(w(G) - t\right)\right]^a} \\
= & \lim_{t \to 0} \left[\frac{1 - G\left(w(G) - tx\right)}{1 - G\left(w(G) - t\right)}\right]^a \\
= & x^{a\beta}
\end{aligned}$$

for every $x < 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left(-(-x)^{a\beta}\right)$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

UNSEEN

Solutions to Question B2

a) Note that $w(F) = \infty$ and

$$\begin{aligned}\frac{\Pr(X = k)}{1 - F(k-1)} &= \frac{a(1-a)^{k-1}}{1 - [1 - (1-a)^{k-1}]} \\ &= \frac{a(1-a)^{k-1}}{(1-a)^{k-1}} \\ &= a.\end{aligned}$$

Hence, there can be no non-degenerate limit.

(4 marks)

UNSEEN

b) Note that $w(F) = 1$ and

$$\begin{aligned}&\frac{\Pr(X = w(F))}{1 - F(w(F)-1)} \\ = \ &\frac{\Pr(X = 1)}{1 - F(1-1)} \\ = \ &\frac{a}{1 - (1-a)} \\ = \ &\frac{a}{a} \\ = \ &1.\end{aligned}$$

Hence, there can be no non-degenerate limit.

(4 marks)

UNSEEN

c) Note that $w(F) = 1$. Note that

$$
\begin{aligned}
\lim_{t \downarrow 0} \frac{1 - F(1 - tx)}{1 - F(1 - t)} &= \lim_{t \downarrow 0} \frac{x f(1 - tx)}{f(1 - t)} \\
&= \lim_{t \downarrow 0} \frac{x (1 - tx)^{a-1} \left[1 - (1 - tx)^a\right]^{b-1}}{(1 - t)^{a-1} \left[1 - (1 - t)^a\right]^{b-1}} \\
&= \lim_{t \downarrow 0} \frac{x \left[1 - (1 - tx)^a\right]^{b-1}}{\left[1 - (1 - t)^a\right]^{b-1}} \\
&= \lim_{t \downarrow 0} \frac{x \left[1 - (1 - atx)\right]^{b-1}}{\left[1 - (1 - at)\right]^{b-1}} \\
&= \lim_{t \downarrow 0} \frac{x \left[atx\right]^{b-1}}{\left[at\right]^{b-1}} \\
&= x^b.
\end{aligned}
$$

So, $F(x)$ belongs to the Weibull domain of attraction.

(4 marks)

UNSEEN

d) Note that $w(F) = \infty$. Then

$$
\begin{aligned}
\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} &= \lim_{t \to \infty} \frac{1 - \left[1 + (tx)^{-a}\right]^{-b}}{1 - \left[1 + t^{-a}\right]^{-b}} \\
&= \lim_{t \to \infty} \frac{1 - \left[1 - b(tx)^{-a}\right]}{1 - \left[1 - bt^{-a}\right]} \\
&= \lim_{t \to \infty} \frac{b(tx)^{-a}}{bt^{-a}} \\
&= x^{-a}.
\end{aligned}
$$

So, $F(x)$ belongs to the Fréchet domain of attraction.

(4 marks)

UNSEEN

e) Note that $w(F) = 1$. Then

$$\begin{aligned}
\lim_{t\to 0} \frac{1 - F(1 - tx)}{1 - F(1 - t)} &= \lim_{t\to 0} \frac{xf(1 - tx)}{f(1 - t)} \\
&= \lim_{t\to 0} \frac{x\frac{C}{\sqrt{(1-tx)tx}}}{\frac{C}{\sqrt{(1-t)t}}} \\
&= \lim_{t\to 0} \frac{x\frac{1}{\sqrt{tx}}}{\frac{1}{\sqrt{t}}} \\
&= \sqrt{x}.
\end{aligned}$$

So, the cdf $F(x)$ belongs to the Weibull domain of attraction.

(4 marks)

UNSEEN

Solutions to Question B3

(a) If X is an absolutely continuous random variable with cdf $F(\cdot)$ then

$$\mathrm{VaR}_p(X) = F^{-1}(p)$$

and

$$\mathrm{ES}_p(X) = \frac{1}{p}\int_0^p F^{-1}(v)dv.$$

(2 marks)

SEEN

(b) (i) The pdf of S is

$$\begin{aligned}
f_S(s) &= \int_{x_1+\cdots+x_k=s} f\left(x_1, x_2, \ldots, x_k\right) dx_k \cdots dx_2 dx_1 \\
&= \int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{k-2}} f\left(x_1, x_2, \ldots, s-x_1-\cdots-x_{k-1}\right) dx_{k-1} \cdots dx_2 dx_1 \\
&= \int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{k-2}} \frac{\Gamma\left(k+\frac{a}{2}\right)}{\Gamma\left(\frac{a}{2}\right)} (1-s)^{\frac{a}{2}-1} dx_{k-1} \cdots dx_2 dx_1 \\
&= \frac{\Gamma\left(k+\frac{a}{2}\right)}{\Gamma\left(\frac{a}{2}\right)} (1-s)^{\frac{a}{2}-1} \int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{k-2}} dx_{k-1} \cdots dx_2 dx_1 \\
&= \frac{\Gamma\left(k+\frac{a}{2}\right)}{\Gamma\left(\frac{a}{2}\right)} (1-s)^{\frac{a}{2}-1} \frac{s^{k-1}}{(k-1)!} \\
&= \frac{1}{B\left(\frac{a}{2}, k\right)} s^{k-1} (1-s)^{\frac{a}{2}-1}.
\end{aligned}$$

(6 marks)

UNSEEN

(b) (ii) The nth moment of S is

$$E\left(S^n\right) = \frac{1}{B\left(\frac{a}{2}, k\right)} \int_0^1 s^{n+k-1} (1-s)^{\frac{a}{2}-1} ds = \frac{1}{B\left(\frac{a}{2}, k\right)} B\left(\frac{a}{2}, n+k\right).$$

(3 marks)

UNSEEN

(b) (iii) The cdf of S is

$$F_S(s) = \frac{1}{B\left(\frac{a}{2}, k\right)} \int_0^s x^{k-1} (1-x)^{\frac{a}{2}-1} \, dx = \frac{1}{B\left(\frac{a}{2}, k\right)} B_x \left(k, \frac{a}{2}\right) = I_x \left(k, \frac{a}{2}\right).$$

(3 marks)

UNSEEN

(b) (iv) Inverting

$$I_x \left(k, \frac{a}{2}\right) = p,$$

we obtain

$$\mathrm{VaR}(S) = I_p^{-1} \left(k, \frac{a}{2}\right).$$

(3 marks)

UNSEEN

(b) (v) The corresponding ES is

$$\mathrm{ES}(S) = \frac{1}{p} \int_0^p I_u^{-1} \left(k, \frac{a}{2}\right) du.$$

(3 marks)

UNSEEN

Solutions to Question B4

a) Note that

$$\begin{aligned}
F_U(u) &= \Pr(U \leq u) \\
&= \Pr\left(\max\left(X_1, X_2, \ldots, X_k\right) \leq u\right) \\
&= \Pr\left(X_1 \leq u, X_2 \leq u, \ldots, X_k \leq u\right) \\
&= F\left(u, u, \ldots, u\right) \\
&= \frac{1}{1 + \displaystyle\sum_{i=1}^{k} \exp\left(-u\right)} \\
&= \frac{1}{1 + k \exp\left(-u\right)}
\end{aligned}$$

for $-\infty < u < \infty$.

(6 marks)

UNSEEN

b) The corresponding pdf is

$$f_U(u) = \frac{k \exp(-u)}{\left[1 + k \exp\left(-u\right)\right]^2}$$

for $-\infty < u < \infty$.

(2 marks)

UNSEEN

c) The moment generating function of T is

$$M_T(s) = k \int_{-\infty}^{\infty} \frac{\exp(su - u)}{\left[1 + k \exp\left(-u\right)\right]^2} du = k^s \int_0^1 x^s (1 - x)^{-s} dx = k^s B(1 + s, 1 - s),$$

where $x = 1 / \left[1 + k \exp(-u)\right]$.

(5 marks)

UNSEEN

d) Inverting

$$\frac{1}{1 + k \exp\left(-u\right)} = p$$

gives

$$\mathrm{VaR}_p(U) = -\log(1-p) + \log k + \log p.$$

(2 marks)

UNSEEN

e) The expected shortfall is

$$\begin{aligned}
\mathrm{ES}_p(U) &= \frac{1}{p}\int_0^p \left[-\log(1-u) + \log k + \log u\right] du \\
&= \log k + \frac{1}{p}\int_0^p \left[-\log(1-u) + \log u\right] du \\
&= \log k + \frac{1}{p}\left\{\left[-u\log(1-u) + u\log u\right]_0^p - \int_0^p \left(\frac{u}{1-u} + 1\right) du\right\} \\
&= \log k + \frac{1}{p}\left\{-p\log(1-p) + p\log p - \int_0^p \left(\frac{u-1+1}{1-u} + 1\right) du\right\} \\
&= \log k + \frac{1}{p}\left\{-p\log(1-p) + p\log p - \int_0^p \frac{1}{1-u} du\right\} \\
&= \log k + \frac{1}{p}\left\{-p\log(1-p) + p\log p - \left[-\log(1-u)\right]_0^p\right\} \\
&= \log k + \frac{1}{p}\left\{-p\log(1-p) + p\log p + \log(1-p)\right\}.
\end{aligned}$$

(5 marks)

UNSEEN

Solutions to Question B5

a) Note that

$$\begin{aligned}
F_V(v) &= \Pr(V \le v) \\
&= 1 - \Pr(V > v) \\
&= 1 - \Pr\left(\min\left(X_1, X_2, \ldots, X_k\right) > v\right) \\
&= 1 - \Pr\left(X_1 > v, X_2 > v, \ldots, X_k > v\right) \\
&= 1 - \overline{F}\left(v, v, \ldots, v\right) \\
&= 1 - \exp\left[-\sum_{i=1}^{k} v - \lambda \max\left(v, v, \ldots, v\right)\right] \\
&= 1 - \exp\left[-kv - \lambda v\right]
\end{aligned}$$

for $v > 0$.

(6 marks)

UNSEEN

b) The corresponding pdf is

$$f_V(v) = (k + \lambda) \exp\left[-kv - \lambda v\right]$$

for $v > 0$.

(2 marks)

UNSEEN

c) Inverting

$$1 - \exp\left[-kv - \lambda v\right] = p$$

gives

$$\mathrm{VaR}_p(V) = -\frac{\log(1-p)}{k + \lambda}.$$

(2 marks)

UNSEEN

d) The expected shortfall is

$$
\begin{aligned}
\mathrm{ES}_p(V) &= -\frac{1}{p(k+\lambda)} \int_0^p \log(1-u) du \\
&= -\frac{1}{p(k+\lambda)} \left\{ [u \log(1-u)]_0^p + \int_0^p \frac{u}{1-u} du \right\} \\
&= -\frac{1}{p(k+\lambda)} \left\{ p \log(1-p) + \int_0^p \frac{u-1+1}{1-u} du \right\} \\
&= -\frac{1}{p(k+\lambda)} \left\{ p \log(1-p) + [-u - \log(1-u)]_0^p \right\} \\
&= -\frac{1}{p(k+\lambda)} \left\{ p \log(1-p) - p - \log(1-p) \right\}.
\end{aligned}
$$

(2 marks)

UNSEEN

e) The likelihood function is

$$
L(\lambda) = \prod_{i=1}^n \left\{ (k+\lambda) \exp\left[-k v_i - \lambda v_i\right] \right\} = (k+\lambda)^n \exp\left[-(k+\lambda) \sum_{i=1}^n v_i \right].
$$

Its log is

$$
\log L(\lambda) = n \log(k+\lambda) - (k+\lambda) \sum_{i=1}^n v_i.
$$

The derivative with respect to λ is

$$
\frac{d \log L(\lambda)}{d\lambda} = \frac{n}{k+\lambda} - \sum_{i=1}^n v_i.
$$

Setting to zero and solving for λ, we obtain

$$
\widehat{\lambda} = \frac{n}{\sum_{i=1}^n v_i} - k.
$$

This is an MLE since

$$
\frac{d^2 \log L(\lambda)}{d\lambda^2} = -\frac{n}{(k+\lambda)^2} < 0.
$$

(8 marks)

UNSEEN