

**SOLUTIONS TO
MATH48181
EXTREME VALUES
AND FINANCIAL RISK EXAM**

Solutions to Question A1

a) The joint cdf of X and Y is

$$\begin{aligned} F_{X,Y}(x, y) &= 1 - \bar{F}_{X,Y}(x, 0) - \bar{F}_{X,Y}(0, y) + \bar{F}_{X,Y}(x, y) \\ &= 1 - (1 + ax)^{-a} - (1 + by)^{-a} + (1 + ax + by + cxy)^{-a}. \end{aligned}$$

(3 marks)

UNSEEN

b) The marginal cdfs of X and Y are

$$F_X(x) = F_{X,Y}(x, \infty) = 1 - (1 + ax)^{-a}$$

and

$$F_Y(y) = F_{X,Y}(\infty, y) = 1 - (1 + by)^{-a}.$$

(2 marks)

UNSEEN

c) First note that $w(F_X) = \infty$. F_X belongs to the Fréchet max domain of attraction since

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1 - F_X(tx)}{1 - F_X(t)} &= \lim_{t \rightarrow \infty} \frac{1 - [1 - (1 + atx)^{-a}]}{1 - [1 - (1 + at)^{-a}]} \\ &= \lim_{t \rightarrow \infty} \frac{(1 + atx)^{-a}}{(1 + at)^{-a}} \\ &= \lim_{t \rightarrow \infty} \left(\frac{1 + atx}{1 + at} \right)^{-a} \\ &= \lim_{t \rightarrow \infty} \left(\frac{\frac{1}{t} + ax}{\frac{1}{t} + a} \right)^{-a} \\ &= \left(\frac{0 + ax}{0 + a} \right)^{-a} \\ &= x^{-a}. \end{aligned}$$

(2 marks)

UNSEEN

d) First note that $w(F_Y) = \infty$. F_Y belongs to the Fréchet max domain of attraction since

$$\begin{aligned}
 \lim_{t \rightarrow \infty} \frac{1 - F_Y(ty)}{1 - F_Y(t)} &= \lim_{t \rightarrow \infty} \frac{1 - [1 - (1 + bty)^{-q}]}{1 - [1 - (1 + bt)^{-q}]} \\
 &= \lim_{t \rightarrow \infty} \frac{(1 + bty)^{-q}}{(1 + bt)^{-q}} \\
 &= \lim_{t \rightarrow \infty} \left(\frac{1 + bty}{1 + bt} \right)^{-q} \\
 &= \lim_{t \rightarrow \infty} \left(\frac{\frac{1}{t} + by}{\frac{1}{t} + b} \right)^{-q} \\
 &= \left(\frac{0 + by}{0 + b} \right)^{-q} \\
 &= y^{-q}.
 \end{aligned}$$

(2 marks)

UNSEEN

e) Use the formulas $a_n = F_X^{-1}\left(1 - \frac{1}{n}\right)$ and $b_n = 0$. Inverting

$$F_X(x) = 1 - (1 + ax)^{-q} = p$$

gives

$$x = \frac{(1 - p)^{-1/q} - 1}{a}.$$

So,

$$F_X^{-1}\left(1 - \frac{1}{n}\right) = \frac{n^{1/q} - 1}{a}.$$

Hence,

$$a_n = \frac{n^{1/q} - 1}{a}$$

and $b_n = 0$.

(2 marks)

UNSEEN

f) Use the formulas $c_n = F_Y^{-1}\left(1 - \frac{1}{n}\right)$ and $d_n = 0$. Inverting

$$F_Y(y) = 1 - (1 + by)^{-q} = p$$

gives

$$y = \frac{(1-p)^{-1/q} - 1}{b}.$$

So,

$$F_Y^{-1}\left(1 - \frac{1}{n}\right) = \frac{n^{1/q} - 1}{b}.$$

Hence,

$$c_n = \frac{n^{1/q} - 1}{b}$$

and $b_n = 0$.

(2 marks)

UNSEEN

g) The limiting cdf is

$$\begin{aligned} & \lim_{n \rightarrow \infty} F_{X,Y}^n(a_n x + b_n, c_n y + d_n) \\ &= \lim_{n \rightarrow \infty} \left[1 - (1 + a_n a x)^{-q} - (1 + c_n b y)^{-q} + (1 + a_n a x + c_n b y + c a_n c_n x y)^{-q} \right]^n \\ &= \lim_{n \rightarrow \infty} \left\{ 1 - [1 + (n^{1/q} - 1)x]^{-q} - [1 + (n^{1/q} - 1)y]^{-q} \right. \\ &\quad \left. + \left[1 + (n^{1/q} - 1)x + (n^{1/q} - 1)y + \frac{c}{ab} (n^{1/q} - 1)^2 xy \right]^{-q} \right\}^n \\ &= \lim_{n \rightarrow \infty} \left\{ 1 - \frac{1}{n} [x + (1-x)n^{-1/q}]^{-q} - \frac{1}{n} [y + (1-y)n^{-1/q}]^{-q} \right. \\ &\quad \left. + \frac{1}{n^2} \left[n^{-2/q} + (n^{-1/q} - n^{-2/q})x + (n^{-1/q} - n^{-2/q})y + \frac{c}{ab} (1 - n^{-1/q})^2 xy \right]^{-q} \right\}^n \\ &= \lim_{n \rightarrow \infty} \left\{ 1 - \frac{1}{n} x^{-q} - \frac{1}{n} y^{-q} + \frac{1}{n^2} \left[\frac{c}{ab} xy \right]^{-q} \right\}^n \\ &= \lim_{n \rightarrow \infty} \left\{ 1 - \frac{1}{n} x^{-q} - \frac{1}{n} y^{-q} \right\}^n \\ &= \exp(-x^{-q} - y^{-q}). \end{aligned}$$

(5 marks)

UNSEEN

h) Yes, the extremes are completely independent since

$$\begin{aligned}\lim_{n \rightarrow \infty} F_{X,Y}^n(a_n x + b_n, c_n y + d_n) &= \exp(-x^{-q} - y^{-q}) \\ &= \exp(-x^{-q}) \exp(-y^{-q}). \\ &= \lim_{n \rightarrow \infty} F_X^n(a_n x + b_n) \lim_{n \rightarrow \infty} F_Y^n(c_n y + d_n).\end{aligned}$$

(2 marks)

UNSEEN

Solutions to Question A2

$C(u_1, u_2)$ is a valid copula if

$$C(u, 0) = 0,$$

$$C(0, u) = 0,$$

$$C(1, u) = u,$$

$$C(u, 1) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) \geq 0.$$

(4 marks)

SEEN

a) for the copula defined by $C(u_1, u_2) = u_1^{1-a} u_2^{1-b} \min(u_1^a, u_2^b)$, we have

$$C(u, 0) = u^{1-a} 0^{1-b} \min(u^a, 0^b) = 0,$$

$$C(0, u) = 0^{1-a} u^{1-b} \min(0^a, u^b) = 0,$$

$$C(1, u) = 1^{1-a} u^{1-b} \min(1^a, u^b) = u,$$

$$C(u, 1) = u^{1-a} 1^{1-b} \min(u^a, 1^b) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \frac{\partial}{\partial u_1} \begin{cases} u_1 u_2^{1-b}, & \text{if } u_1^a \leq u_2^b, \\ u_1^{1-a} u_2, & \text{if } u_1^a > u_2^b \end{cases} = \begin{cases} u_2^{1-b}, & \text{if } u_1^a \leq u_2^b, \\ (1-a)u_1^{-a} u_2, & \text{if } u_1^a > u_2^b \end{cases} \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \frac{\partial}{\partial u_2} \begin{cases} u_1 u_2^{1-b}, & \text{if } u_1^a \leq u_2^b, \\ u_1^{1-a} u_2, & \text{if } u_1^a > u_2^b \end{cases} = \begin{cases} (1-b)u_1 u_2^{-b}, & \text{if } u_1^a \leq u_2^b, \\ u_1^{1-a}, & \text{if } u_1^a > u_2^b \end{cases} \geq 0.$$

(4 marks)

UNSEEN

b) for the copula defined by $C(u_1, u_2) = \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2}$, we have

$$C(u, 0) = \frac{u \cdot 0}{u + 0 - u \cdot 0} = 0,$$

$$C(0, u) = \frac{0 \cdot u}{0 + u - 0 \cdot u} = 0,$$

$$C(1, u) = \frac{1 \cdot u}{1 + u - 1 \cdot u} = u,$$

$$C(u, 1) = \frac{u \cdot 1}{u + 1 - u \cdot 1} = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \frac{u_2^2}{(u_1 + u_2 - u_1 u_2)^2} \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \frac{u_1^2}{(u_1 + u_2 - u_1 u_2)^2} \geq 0.$$

(4 marks)

UNSEEN

c) for the copula defined by $C(u_1, u_2) = [C_1(u_1, u_2) C_2(u_1, u_2) \cdots C_p(u_1, u_2)]^{1/p}$, we have

$$C(u, 0) = [C_1(u, 0) C_2(u, 0) \cdots C_p(u, 0)]^{1/p} = 0,$$

$$C(0, u) = [C_1(0, u) C_2(0, u) \cdots C_p(0, u)]^{1/p} = 0,$$

$$C(1, u) = [C_1(1, u) C_2(1, u) \cdots C_p(1, u)]^{1/p} = [u^p]^{1/p} = u,$$

$$C(u, 1) = [C_1(u, 1) C_2(u, 1) \cdots C_p(u, 1)]^{1/p} = [u^p]^{1/p} = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \frac{1}{p} [C_1(u_1, u_2) C_2(u_1, u_2) \cdots C_p(u_1, u_2)]^{\frac{1}{p}-1} \sum_{k=1}^p \left[\sum_{j=1, j \neq k}^p C_j(u_1, u_2) \right] \frac{\partial}{\partial u_1} C_k(u_1, u_2) \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \frac{1}{p} [C_1(u, 1) C_2(u, 1) \cdots C_p(u, 1)]^{\frac{1}{p}-1} \sum_{k=1}^p \left[\sum_{j=1, j \neq k}^p C_j(u_1, u_2) \right] \frac{\partial}{\partial u_2} C_k(u_1, u_2) \geq 0.$$

(4 marks)

UNSEEN

d) for the copula defined by $C(u_1, u_2) = w_1 C_1(u_1, u_2) + w_2 C_2(u_1, u_2) + \cdots + w_p C_p(u_1, u_2)$, we have

$$C(u, 0) = w_1 C_1(u, 0) + w_2 C_2(u, 0) + \cdots + w_p C_p(u, 0),$$

$$C(0, u) = w_1 C_1(0, u) + w_2 C_2(0, u) + \cdots + w_p C_p(0, u),$$

$$C(1, u) = w_1 C_1(1, u) + w_2 C_2(1, u) + \cdots + w_p C_p(1, u) = (w_1 + w_2 + \cdots + w_p) u = u,$$

$$C(u, 1) = w_1 C_1(u, 1) + w_2 C_2(u, 1) + \cdots + w_p C_p(u, 1) = (w_1 + w_2 + \cdots + w_p) u = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \sum_{k=1}^p w_k \frac{\partial}{\partial u_1} C_k(u_1, u_2) \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \sum_{k=1}^p w_k \frac{\partial}{\partial u_2} C_k(u_1, u_2) \geq 0.$$

(4 marks)

UNSEEN

Solutions to Question A3

a) We can write

$$\bar{G}(x, y) = \exp \left[-(x + y)A \left(\frac{y}{x + y} \right) \right]$$

for $x > 0$ and $y > 0$, where

$$A(w) = 1 - \alpha w(1 - w) [1 - \beta w(1 - w)].$$

We now check the conditions for $A(\cdot)$. Clearly,

$$A(0) = 1$$

and

$$A(1) = 1.$$

Also $A(w) \leq 1$ since $\alpha w(1 - w) [1 - \beta w(1 - w)] \geq 0$ which holds since $w(1 - w) \leq 1/4$ for all $0 \leq w \leq 1$.

Also

$$\begin{aligned} A(w) &\geq w \\ \iff 1 - \alpha w(1 - w) [1 - \beta w(1 - w)] &\geq w \\ \iff 1 - w - \alpha w(1 - w) [1 - \beta w(1 - w)] &\geq 0 \\ \iff 1 - \alpha w [1 - \beta w(1 - w)] &\geq 0 \\ \iff 1 - \alpha w + \alpha \beta w^2 - \alpha \beta w^3 = g(w) &\text{ say } \geq 0. \end{aligned}$$

Note that $g'(w) = -\alpha + 2\alpha\beta w - 3\alpha\beta w^2$ and $g''(w) = 2\alpha\beta - 6\alpha\beta w = 2\alpha\beta(1 - 3w)$. Since $g'(0) = -\alpha$ and $g'(1) = -\alpha - \alpha\beta$, $g(w)$ is a decreasing function for $0 \leq w \leq 1$. Because $g(1) = 1 - \alpha \geq 0$, $g(w) \geq 0$ for all $0 \leq w \leq 1$.

Also

$$\begin{aligned} A(w) &\geq 1 - w \\ \iff 1 - \alpha w(1 - w) [1 - \beta w(1 - w)] &\geq 1 - w \\ \iff w - \alpha w(1 - w) [1 - \beta w(1 - w)] &\geq 0 \\ \iff 1 - \alpha(1 - w) [1 - \beta w(1 - w)] &\geq 0 \\ \iff 1 - \alpha(1 - w) + \alpha\beta(1 - w)^2 - \alpha\beta(1 - w)^3 = g(w) &\text{ say } \geq 0. \end{aligned}$$

Note that $g'(w) = \alpha - 2\alpha\beta(1 - w) + 3\alpha\beta(1 - w)^2$ and $g''(w) = 2\alpha\beta - 6\alpha\beta(1 - w) = 2\alpha\beta(3w - 1)$. Since $g'(0) = \alpha + \alpha\beta$ and $g'(1) = \alpha$, $g(w)$ is an increasing function for $0 \leq w \leq 1$. Because $g(0) = 1 - \alpha \geq 0$, $g(w) \geq 0$ for all $0 \leq w \leq 1$.

The first and second derivatives of $A(w)$ are

$$A'(w) = -\alpha + 2\alpha w + 2\alpha\beta w - 6\alpha\beta w^2 + 4\alpha\beta w^3$$

and

$$A''(w) = 2\alpha + 2\alpha\beta - 12\alpha\beta w + 12\alpha\beta w^2,$$

respectively. Note that

$$A'''(w) = -12\alpha\beta + 24\alpha\beta w = 12\alpha\beta(2w - 1).$$

Since

$$A''(1/2) = 2\alpha + 2\alpha\beta - 6\alpha\beta w + 3\alpha\beta = \alpha(2 - \beta) \geq 0,$$

we have $A''(w) \geq 0$ for all $0 \leq w \leq 1$. Hence, $A(w)$ is convex.

(7 marks)

UNSEEN

b) Note that

$$\bar{G}(x, 0) = \exp\{-(x+0)A(0)\} = \exp(-x)$$

and

$$\bar{G}(0, y) = \exp\{-(0+y)A(1)\} = \exp(-y).$$

So, the joint cdf is

$$G(x, y) = 1 - \exp(-x) - \exp(-y) + \exp\left\{-x - y + \frac{\alpha xy}{x+y} \left[1 - \frac{\beta xy}{(x+y)^2}\right]\right\}.$$

(1 marks)

UNSEEN

c) the derivative of joint cdf with respect to x is

$$\frac{\partial G(x, y)}{\partial x} = \exp(-x) + \bar{G}(x, y) \left[-1 + \frac{\alpha y^2}{(x+y)^2} - \frac{2\alpha\beta xy^3}{(x+y)^4} + \frac{3\alpha\beta x^2 y^2}{(x+y)^4}\right],$$

so the conditional cdf of Y given $X = x$ is

$$G(y | x) = 1 + \exp\left\{-y + \frac{\alpha xy}{x+y} \left[1 - \frac{\beta xy}{(x+y)^2}\right]\right\} \left[-1 + \frac{\alpha y^2}{(x+y)^2} - \frac{2\alpha\beta xy^3}{(x+y)^4} + \frac{3\alpha\beta x^2 y^2}{(x+y)^4}\right].$$

(4 marks)

UNSEEN

d) the derivative of joint cdf with respect to y is

$$\frac{\partial G(x, y)}{\partial y} = \exp(-x) + \bar{G}(x, y) \left[-1 + \frac{\alpha x^2}{(x+y)^2} - \frac{2\alpha\beta y x^3}{(x+y)^4} + \frac{3\alpha\beta x^2 y^2}{(x+y)^4} \right],$$

so the conditional cdf of Y given $Y = y$ is

$$G(y | x) = 1 + \exp \left\{ -x + \frac{\alpha x y}{x+y} \left[1 - \frac{\beta x y}{(x+y)^2} \right] \right\} \left[-1 + \frac{\alpha x^2}{(x+y)^2} - \frac{2\alpha\beta y x^3}{(x+y)^4} + \frac{3\alpha\beta x^2 y^2}{(x+y)^4} \right].$$

(4 marks)

UNSEEN

e) the derivative of joint cdf with respect to x and y is

$$g(x, y) = \bar{G}(x, y) \left[-1 + \frac{\alpha y^2}{(x+y)^2} - \frac{2\alpha\beta x y^3}{(x+y)^4} + \frac{3\alpha\beta x^2 y^2}{(x+y)^4} \right] \left[-1 + \frac{\alpha x^2}{(x+y)^2} - \frac{2\alpha\beta y x^3}{(x+y)^4} + \frac{3\alpha\beta x^2 y^2}{(x+y)^4} \right] \\ + \bar{G}(x, y) \left[\frac{2\alpha x y}{(x+y)^3} - \frac{2\alpha\beta x y^2 (3x-y)}{(x+y)^5} + \frac{6\alpha\beta x^2 y (x-y)}{(x+y)^5} \right].$$

(4 marks)

UNSEEN

Solutions to Question B1

If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cdf of a normalized version of M_n converges to G , i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n(a_n x + b_n) \rightarrow G(x)$$

as $n \rightarrow \infty$ then G must be of the same type as (cdf's G and G^* are of the same type if $G^*(x) = G(ax + b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$\begin{aligned} I & : \Lambda(x) = \exp\{-\exp(-x)\}, \quad x \in \mathfrak{R}; \\ II & : \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\{-x^{-\alpha}\} & \text{if } x \geq 0 \end{cases} \\ & \quad \text{for some } \alpha > 0; \\ III & : \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^\alpha\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \\ & \quad \text{for some } \alpha > 0. \end{aligned}$$

(4 marks)

SEEN

The necessary and sufficient conditions for the three extreme value distributions are:

$$\begin{aligned} I & : \exists \gamma(t) > 0 \text{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = \exp(-x), \quad x \in \mathfrak{R}, \\ II & : w(F) = \infty \text{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \quad x > 0, \\ III & : w(F) < \infty \text{ and } \lim_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} = x^\alpha, \quad x > 0. \end{aligned}$$

(4 marks)

SEEN

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function say $h(t)$ such that

$$\lim_{t \rightarrow w(G)} \frac{1 - G(t + xh(t))}{1 - G(t)} = e^{-x}$$

for every $x > 0$. But

$$\begin{aligned}
& \lim_{t \rightarrow w(F)} \frac{1 - F(t + xh(t))}{1 - F(t)} \\
= & \lim_{t \rightarrow w(F)} \frac{1 - \frac{[G(t+xh(t))]^a}{[G(t+xh(t))]^a + [1-G(t+xh(t))]^a}}{1 - \frac{[G(t)]^a}{[G(t)]^a + [1-G(t)]^a}} \\
= & \lim_{t \rightarrow w(F)} \frac{\frac{[1-G(t+xh(t))]^a}{[G(t+xh(t))]^a + [1-G(t+xh(t))]^a}}{\frac{[1-G(t)]^a}{[G(t)]^a + [1-G(t)]^a}} \\
= & \lim_{t \rightarrow w(F)} \frac{[1 - G(t + xh(t))]^a}{[1 - G(t)]^a} \frac{[G(t)]^a + [1 - G(t)]^a}{[G(t + xh(t))]^a + [1 - G(t + xh(t))]^a} \\
= & \lim_{t \rightarrow w(G)} \frac{[1 - G(t + xh(t))]^a}{[1 - G(t)]^a} \lim_{t \rightarrow w(G)} \frac{[G(t)]^a + [1 - G(t)]^a}{[G(t + xh(t))]^a + [1 - G(t + xh(t))]^a} \\
= & \lim_{t \rightarrow w(G)} \frac{[1 - G(t + xh(t))]^a}{[1 - G(t)]^a} \frac{1 + 0}{1 + 0} \\
= & \lim_{t \rightarrow w(G)} \frac{[1 - G(t + xh(t))]^a}{[1 - G(t)]^a} \\
= & \lim_{t \rightarrow w(G)} \left[\frac{1 - G(t + xh(t))}{1 - G(t)} \right]^a \\
= & \exp(-ax)
\end{aligned}$$

for every $x > 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n \rightarrow \infty} P \left(\frac{M_n - b_n}{a_n} \leq x \right) = \exp[-\exp(-ax)]$$

for some suitable norming constants $a_n > 0$ and b_n .

(4 marks)

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t \rightarrow \infty} \frac{1 - G(tx)}{1 - G(t)} = x^{-\beta}$$

for every $x > 0$. But

$$\begin{aligned}
& \lim_{t \rightarrow \infty} \frac{1 - F(tx)}{1 - F(t)} \\
&= \lim_{t \rightarrow \infty} \frac{1 - \frac{[G(tx)]^a}{[G(tx)]^a + [1 - G(tx)]^a}}{1 - \frac{[G(t)]^a}{[G(t)]^a + [1 - G(t)]^a}} \\
&= \lim_{t \rightarrow \infty} \frac{\frac{[1 - G(tx)]^a}{[G(tx)]^a + [1 - G(tx)]^a}}{\frac{[1 - G(t)]^a}{[G(t)]^a + [1 - G(t)]^a}} \\
&= \lim_{t \rightarrow \infty} \frac{[1 - G(tx)]^a}{[1 - G(t)]^a} \frac{[G(t)]^a + [1 - G(t)]^a}{[G(tx)]^a + [1 - G(tx)]^a} \\
&= \lim_{t \rightarrow \infty} \frac{[1 - G(tx)]^a}{[1 - G(t)]^a} \lim_{t \rightarrow \infty} \frac{[G(t)]^a + [1 - G(t)]^a}{[G(tx)]^a + [1 - G(tx)]^a} \\
&= \lim_{t \rightarrow \infty} \frac{[1 - G(tx)]^a}{[1 - G(t)]^a} \frac{1 + 0}{1 + 0} \\
&= \lim_{t \rightarrow \infty} \frac{[1 - G(tx)]^a}{[1 - G(t)]^a} \\
&= \lim_{t \rightarrow \infty} \left[\frac{1 - G(tx)}{1 - G(t)} \right]^a \\
&= x^{-a\beta}
\end{aligned}$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n \rightarrow \infty} P \left(\frac{M_n - b_n}{a_n} \leq x \right) = \exp(-x^{-a\beta})$$

for some suitable norming constants $a_n > 0$ and b_n .

(4 marks)

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t \rightarrow 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} = x^\beta$$

for every $x > 0$.

$$\begin{aligned}
& \lim_{t \rightarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} \\
= & \lim_{t \rightarrow 0} \frac{1 - \frac{[G(w(F) - tx)]^\alpha}{[G(w(F) - tx)]^\alpha + [1 - G(w(F) - tx)]^\alpha}}{1 - \frac{[G(w(F) - t)]^\alpha}{[G(w(F) - t)]^\alpha + [1 - G(w(F) - t)]^\alpha}} \\
= & \lim_{t \rightarrow 0} \frac{\frac{[1 - G(w(F) - tx)]^\alpha}{[G(w(F) - tx)]^\alpha + [1 - G(w(F) - tx)]^\alpha}}{\frac{[1 - G(w(F) - t)]^\alpha}{[G(w(F) - t)]^\alpha + [1 - G(w(F) - t)]^\alpha}} \\
= & \lim_{t \rightarrow 0} \frac{[1 - G(w(F) - tx)]^\alpha}{[1 - G(w(F) - t)]^\alpha} \frac{[G(w(F) - t)]^\alpha + [1 - G(w(F) - t)]^\alpha}{[G(w(F) - tx)]^\alpha + [1 - G(w(F) - tx)]^\alpha} \\
= & \lim_{t \rightarrow 0} \frac{[1 - G(w(G) - tx)]^\alpha}{[1 - G(w(G) - t)]^\alpha} \lim_{t \rightarrow 0} \frac{[G(w(G) - t)]^\alpha + [1 - G(w(G) - t)]^\alpha}{[G(w(G) - tx)]^\alpha + [1 - G(w(G) - tx)]^\alpha} \\
= & \lim_{t \rightarrow 0} \frac{[1 - G(w(G) - tx)]^\alpha}{[1 - G(w(G) - t)]^\alpha} \frac{1 + 0}{1 + 0} \\
= & \lim_{t \rightarrow 0} \frac{[1 - G(w(G) - tx)]^\alpha}{[1 - G(w(G) - t)]^\alpha} \\
= & \lim_{t \rightarrow 0} \left[\frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} \right]^\alpha \\
= & x^{a\beta}
\end{aligned}$$

for every $x < 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{n \rightarrow \infty} P \left(\frac{M_n - b_n}{a_n} \leq x \right) = \exp(-(-x)^{a\beta})$$

for some suitable norming constants $a_n > 0$ and b_n .

(4 marks)

UNSEEN

Solutions to Question B2

a) Note that $w(F) = \infty$ and

$$\begin{aligned}\frac{\Pr(X = k)}{1 - F(k - 1)} &= \frac{a(1 - a)^{k-1}}{1 - [1 - (1 - a)^{k-1}]} \\ &= \frac{a(1 - a)^{k-1}}{(1 - a)^{k-1}} \\ &= a.\end{aligned}$$

Hence, there can be no non-degenerate limit.

(4 marks)

UNSEEN

b) Note that $w(F) = 1$ and

$$\begin{aligned}&\frac{\Pr(X = w(F))}{1 - F(w(F) - 1)} \\ &= \frac{\Pr(X = 1)}{1 - F(1 - 1)} \\ &= \frac{a}{1 - (1 - a)} \\ &= \frac{a}{a} \\ &= 1.\end{aligned}$$

Hence, there can be no non-degenerate limit.

(4 marks)

UNSEEN

c) Note that $w(F) = 1$. Note that

$$\begin{aligned}
 \lim_{t \downarrow 0} \frac{1 - F(1 - tx)}{1 - F(1 - t)} &= \lim_{t \downarrow 0} \frac{xf(1 - tx)}{f(1 - t)} \\
 &= \lim_{t \downarrow 0} \frac{x(1 - tx)^{a-1} [1 - (1 - tx)^a]^{b-1}}{(1 - t)^{a-1} [1 - (1 - t)^a]^{b-1}} \\
 &= \lim_{t \downarrow 0} \frac{x [1 - (1 - tx)^a]^{b-1}}{[1 - (1 - t)^a]^{b-1}} \\
 &= \lim_{t \downarrow 0} \frac{x [1 - (1 - atx)]^{b-1}}{[1 - (1 - at)]^{b-1}} \\
 &= \lim_{t \downarrow 0} \frac{x [atx]^{b-1}}{[at]^{b-1}} \\
 &= x^b.
 \end{aligned}$$

So, $F(x)$ belongs to the Weibull domain of attraction.

(4 marks)

UNSEEN

d) Note that $w(F) = \infty$. Then

$$\begin{aligned}
 \lim_{t \rightarrow \infty} \frac{1 - F(tx)}{1 - F(t)} &= \lim_{t \rightarrow \infty} \frac{1 - [1 + (tx)^{-a}]^{-b}}{1 - [1 + t^{-a}]^{-b}} \\
 &= \lim_{t \rightarrow \infty} \frac{1 - [1 - b(tx)^{-a}]^{-1}}{1 - [1 - bt^{-a}]^{-1}} \\
 &= \lim_{t \rightarrow \infty} \frac{b(tx)^{-a}}{bt^{-a}} \\
 &= x^{-a}.
 \end{aligned}$$

So, $F(x)$ belongs to the Fréchet domain of attraction.

(4 marks)

UNSEEN

e) Note that $w(F) = 1$. Then

$$\begin{aligned}\lim_{t \rightarrow 0} \frac{1 - F(1 - tx)}{1 - F(1 - t)} &= \lim_{t \rightarrow 0} \frac{xf(1 - tx)}{f(1 - t)} \\ &= \lim_{t \rightarrow 0} \frac{x \frac{C}{\sqrt{(1-t)tx}}}{\frac{C}{\sqrt{(1-t)t}}} \\ &= \lim_{t \rightarrow 0} \frac{x \frac{1}{\sqrt{tx}}}{\frac{1}{\sqrt{t}}} \\ &= \sqrt{x}.\end{aligned}$$

So, the cdf $F(x)$ belongs to the Weibull domain of attraction.

(4 marks)

UNSEEN

Solutions to Question B3

(a) If X is an absolutely continuous random variable with cdf $F(\cdot)$ then

$$\text{VaR}_p(X) = F^{-1}(p)$$

and

$$\text{ES}_p(X) = \frac{1}{p} \int_0^p F^{-1}(v) dv.$$

(2 marks)

SEEN

(b) (i) The pdf of S is

$$\begin{aligned} f_S(s) &= \int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{k-1}} f(x_1, x_2, \dots, x_k) dx_k \cdots dx_2 dx_1 \\ &= \int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{k-1}} \frac{\Gamma\left(k + \frac{a}{2}\right)}{\Gamma\left(\frac{a}{2}\right)} (1-s)^{\frac{a}{2}-1} dx_k \cdots dx_2 dx_1 \\ &= \frac{\Gamma\left(k + \frac{a}{2}\right)}{\Gamma\left(\frac{a}{2}\right)} (1-s)^{\frac{a}{2}-1} \int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{k-1}} dx_k \cdots dx_2 dx_1 \\ &= \frac{\Gamma\left(k + \frac{a}{2}\right)}{\Gamma\left(\frac{a}{2}\right)} (1-s)^{\frac{a}{2}-1} \frac{s^{k-1}}{(k-1)!} \\ &= \frac{1}{B\left(\frac{a}{2}, k\right)} s^{k-1} (1-s)^{\frac{a}{2}-1}. \end{aligned}$$

(6 marks)

UNSEEN

(b) (ii) The n th moment of S is

$$E(S^n) = \frac{1}{B\left(\frac{a}{2}, k\right)} \int_0^1 s^{n+k-1} (1-s)^{\frac{a}{2}-1} ds = \frac{1}{B\left(\frac{a}{2}, k\right)} B\left(\frac{a}{2}, n+k\right).$$

(3 marks)

UNSEEN

(b) (iii) The cdf of S is

$$F_S(s) = \frac{1}{B\left(\frac{a}{2}, k+1\right)} \int_0^s x^{k-1} (1-x)^{\frac{a}{2}-1} dx = \frac{1}{B\left(\frac{a}{2}, k\right)} B_x\left(k, \frac{a}{2}\right) = I_x\left(k, \frac{a}{2}\right).$$

(3 marks)

UNSEEN

(b) (iv) Inverting

$$I_x\left(k, \frac{a}{2}\right) = p,$$

we obtain

$$\text{VaR}(S) = I_p^{-1}\left(k, \frac{a}{2}\right).$$

(3 marks)

UNSEEN

(b) (v) The corresponding ES is

$$\text{ES}(S) = \frac{1}{p} \int_0^p I_u^{-1}\left(k, \frac{a}{2}\right) du.$$

(3 marks)

UNSEEN

Solutions to Question B4

a) Note that

$$\begin{aligned}F_U(u) &= \Pr(U \leq u) \\&= \Pr(\max(X_1, X_2, \dots, X_k) \leq u) \\&= \Pr(X_1 \leq u, X_2 \leq u, \dots, X_k \leq u) \\&= F(u, u, \dots, u) \\&= \frac{1}{1 + \sum_{i=1}^k \exp(-u)} \\&= \frac{1}{1 + k \exp(-u)}\end{aligned}$$

for $-\infty < u < \infty$.

(6 marks)

UNSEEN

b) The corresponding pdf is

$$f_U(u) = \frac{k \exp(-u)}{[1 + k \exp(-u)]^2}$$

for $-\infty < u < \infty$.

(2 marks)

UNSEEN

c) The moment generating function of T is

$$M_T(s) = k \int_{-\infty}^{\infty} \frac{\exp(su - u)}{[1 + k \exp(-u)]^2} du = k^s \int_0^1 x^s (1 - x)^{-s} dx = k^s B(1 + s, 1 - s),$$

where $x = 1/[1 + k \exp(-u)]$.

(5 marks)

UNSEEN

d) Inverting

$$\frac{1}{1 + k \exp(-u)} = p$$

gives

$$\text{VaR}_p(U) = -\log(1-p) + \log k + \log p.$$

(2 marks)

UNSEEN

e) The expected shortfall is

$$\begin{aligned} \text{ES}_p(U) &= \frac{1}{p} \int_0^p [-\log(1-u) + \log k + \log u] du \\ &= \log k + \frac{1}{p} \int_0^p [-\log(1-u) + \log u] du \\ &= \log k + \frac{1}{p} \left\{ [-u \log(1-u) + u \log u]_0^p - \int_0^p \left(\frac{u}{1-u} + 1 \right) du \right\} \\ &= \log k + \frac{1}{p} \left\{ -p \log(1-p) + p \log p - \int_0^p \left(\frac{u-1+1}{1-u} + 1 \right) du \right\} \\ &= \log k + \frac{1}{p} \left\{ -p \log(1-p) + p \log p - \int_0^p \frac{1}{1-u} du \right\} \\ &= \log k + \frac{1}{p} \left\{ -p \log(1-p) + p \log p - [-\log(1-u)]_0^p \right\} \\ &= \log k + \frac{1}{p} \left\{ -p \log(1-p) + p \log p + \log(1-p) \right\}. \end{aligned}$$

(5 marks)

UNSEEN

Solutions to Question B5

a) Note that

$$\begin{aligned}F_V(v) &= \Pr(V \leq v) \\&= 1 - \Pr(V > v) \\&= 1 - \Pr(\min(X_1, X_2, \dots, X_k) > v) \\&= 1 - \Pr(X_1 > v, X_2 > v, \dots, X_k > v) \\&= 1 - \bar{F}(v, v, \dots, v) \\&= 1 - \exp\left[-\sum_{i=1}^k v - \lambda \max(v, v, \dots, v)\right] \\&= 1 - \exp[-kv - \lambda v]\end{aligned}$$

for $v > 0$.

(6 marks)

UNSEEN

b) The corresponding pdf is

$$f_V(v) = (k + \lambda) \exp[-kv - \lambda v]$$

for $v > 0$.

(2 marks)

UNSEEN

c) Inverting

$$1 - \exp[-kv - \lambda v] = p$$

gives

$$\text{VaR}_p(V) = -\frac{\log(1-p)}{k + \lambda}.$$

(2 marks)

UNSEEN

d) The expected shortfall is

$$\begin{aligned}
 \text{ES}_p(V) &= -\frac{1}{p(k+\lambda)} \int_0^p \log(1-u) du \\
 &= -\frac{1}{p(k+\lambda)} \left\{ [u \log(1-u)]_0^p + \int_0^p \frac{u}{1-u} du \right\} \\
 &= -\frac{1}{p(k+\lambda)} \left\{ p \log(1-p) + \int_0^p \frac{u-1+1}{1-u} du \right\} \\
 &= -\frac{1}{p(k+\lambda)} \{ p \log(1-p) + [-u - \log(1-u)]_0^p \} \\
 &= -\frac{1}{p(k+\lambda)} \{ p \log(1-p) - p - \log(1-p) \}.
 \end{aligned}$$

(2 marks)

UNSEEN

e) The likelihood function is

$$L(\lambda) = \prod_{i=1}^n \{(k+\lambda) \exp[-kv_i - \lambda v_i]\} = (k+\lambda)^n \exp\left[-(k+\lambda) \sum_{i=1}^n v_i\right].$$

Its log is

$$\log L(\lambda) = n \log(k+\lambda) - (k+\lambda) \sum_{i=1}^n v_i.$$

The derivative with respect to λ is

$$\frac{d \log L(\lambda)}{d\lambda} = \frac{n}{k+\lambda} - \sum_{i=1}^n v_i.$$

Setting to zero and solving for λ , we obtain

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^n v_i} - k.$$

This is an MLE since

$$\frac{d^2 \log L(\lambda)}{d\lambda^2} = -\frac{n}{(k+\lambda)^2} < 0.$$

(8 marks)

UNSEEN