SOLUTIONS TO
MATH10802
INTRO TO STATISTICS

Solutions to Question 1

ILOs addressed: present numerical summaries of a data set.

Suppose that we have the following sample of observations

-1.3, -0.59, 0.1, -1.4, -0.22, -0.35, -0.76, -0.2, 0.41, 0.32

The sample mean is

$$\frac{1}{10}(-1.3 - 0.59 + 0.1 - 1.4 - 0.22 - 0.35 - 0.76 - 0.2 + 0.41 + 0.32) = -0.399$$

(1 marks)

UNSEEN

The sample variance is

$$\frac{1}{9}\left[(-1.3 - \overline{x})^2 + (-0.59 - \overline{x})^2 + \cdots + (0.32 - \overline{x})^2\right] = 0.3861211$$

(1 marks)

UNSEEN

Arrange the data as

-1.40, -1.30, -0.76, -0.59, -0.35, -0.22, -0.20, 0.10, 0.32, 0.41

The middle two numbers are -0.35 and -0.22. The median is their average which is -0.285. (1 marks)

UNSEEN

Note that $r = 2.75$ and $r' = 3$, so $Q(1/4) = x_{(2)} + 0.75\left(x_{(3)} - x_{(2)}\right) = -0.895$. (1 marks)

UNSEEN

Note that $r = 8.25$ and $r' = 8$, so $Q(3/4) = x_{(8)} + 0.25\left(x_{(9)} - x_{(8)}\right) = 0.155$. (1 marks)

UNSEEN

The sample quartile range is $0.10 - (-0.76) = 0.86$. (1 marks)

UNSEEN

The range of the data are

0.41 - (-1.4) = 1.81.

(1 marks)

UNSEEN

$-0.76 - 1.5 \times 0.86 = -2.05$ and $0.1 + 1.5 \times 0.86 = 1.39$, so there are no outliers (1 marks)

UNSEEN

since mean $\neq$ median the data are not symmetrically distributed. (1 marks)

UNSEEN

since mean $<$ median the data are skewed to the left. (1 marks)

UNSEEN

Solutions to Question 2

ILOs addressed: define elementary statistical concepts and terminology such as unbiasedness; analyse and compare statistical properties of simple estimators.

(a) Suppose $\widehat{\theta}$ is an estimator of θ based on a random sample of size n. Define what is meant by the following:

(i) $\widehat{\theta}$ is an unbiased estimator of θ if $E\left(\widehat{\theta}\right) = \theta$; (1 marks)

(ii) the bias of $\widehat{\theta}$ is $E\left(\widehat{\theta}\right) - \theta$; (1 marks)

(iii) the mean squared error of $\widehat{\theta}$ is $E\left[\left(\widehat{\theta} - \theta\right)^2\right]$; (1 marks)

(iv) $\widehat{\theta}$ is a consistent estimator of θ if $\lim_{n\to\infty} E\left[\left(\widehat{\theta} - \theta\right)^2\right] = 0$. (1 marks)

UP TO THIS BOOK WORK.

(b) Suppose $X_1, \ldots, X_n$ are independent Uniform$[0, \theta]$ random variables. Let $\widehat{\theta_1} = \frac{2(X_1+\cdots+X_n)}{n}$ and $\widehat{\theta_2} = \max\left(X_1, \ldots, X_n\right)$ denote possible estimators of θ.

(i) The bias and mean squared error of $\widehat{\theta_1}$ are

$$
\begin{aligned}
\text{bias}\left(\widehat{\theta_1}\right) &= E\left(\widehat{\theta_1}\right) - \theta \\
&= \frac{2}{n} E\left(X_1 + \cdots + X_n\right) - \theta \\
&= \frac{2}{n} \left[E\left(X_1\right) + \cdots + E\left(X_n\right)\right] - \theta \\
&= \frac{2}{n} \left[\frac{\theta}{2} + \cdots + \frac{\theta}{2}\right] - \theta \\
&= \theta - \theta \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
\text{MSE}\left(\widehat{\theta_1}\right) &= \text{Var}\left(\widehat{\theta_1}\right) \\
&= \frac{4}{n^2} \text{Var}\left(X_1 + \cdots + X_n\right) \\
&= \frac{4}{n^2} \left[\text{Var}\left(X_1\right) + \cdots + \text{Var}\left(X_n\right)\right] \\
&= \frac{4}{n^2} \left[\frac{\theta^2}{12} + \cdots + \frac{\theta^2}{12}\right] \\
&= \frac{\theta^2}{3n}.
\end{aligned}
$$

(2 marks)

UNSEEN

(ii) Let $Z = \widehat{\theta_2}$. The cdf and the pdf of Z are

$$\begin{aligned} F_Z(z) &= \Pr\left(\max\left(X_1, \ldots, X_n\right) \leq z\right) \\ &= \Pr\left(X_1 \leq z, \ldots, X_n \leq z\right) \\ &= \Pr\left(X_1 \leq z\right) \cdots \Pr\left(X_n \leq z\right) \\ &= \frac{z}{\theta} \cdots \frac{z}{\theta} \\ &= \frac{z^n}{\theta^n} \end{aligned}$$

and

$$f_Z(z) = \frac{n z^{n-1}}{\theta^n}.$$

So, the bias and mean squared error of $\widehat{\theta_2}$ are

$$\begin{aligned} \text{bias}\left(\widehat{\theta_2}\right) &= E\left(Z\right) - \theta \\ &= \frac{n}{\theta^n} \int_0^\theta z^n dz - \theta \\ &= \frac{n}{\theta^n} \left[\frac{z^{n+1}}{n+1}\right]_0^\theta - \theta \\ &= \frac{n}{\theta^n} \left[\frac{\theta^{n+1}}{n+1} - 0\right] - \theta \\ &= \frac{n\theta}{n+1} - \theta \\ &= -\frac{\theta}{n+1} \end{aligned}$$

and

$$
\begin{aligned}
\mathrm{MSE}\left(\widehat{\theta}_1\right) &= \mathrm{Var}\left(Z\right) + \left(-\frac{\theta}{n+1}\right)^2 \\
&= E\left(Z^2\right) - E^2\left(Z\right) + \frac{\theta^2}{(n+1)^2} \\
&= E\left(Z^2\right) - \frac{n^2\theta^2}{(n+1)^2} + \frac{\theta^2}{(n+1)^2} \\
&= \frac{n}{\theta^n}\int_0^\theta z^{n+1}dz - \frac{n^2\theta^2}{(n+1)^2} + \frac{\theta^2}{(n+1)^2} \\
&= \frac{n}{\theta^n}\left[\frac{z^{n+2}}{n+2}\right]_0^\theta - \frac{n^2\theta^2}{(n+1)^2} + \frac{\theta^2}{(n+1)^2} \\
&= \frac{n\theta^2}{n+2} - \frac{n^2\theta^2}{(n+1)^2} + \frac{\theta^2}{(n+1)^2} \\
&= \frac{2\theta^2}{(n+1)(n+2)}.
\end{aligned}
$$

(2 marks)

UNSEEN

(iii) $\widehat{\theta}_1$ is better with respect to bias since bias $\widehat{\theta}_1 = 0$ and bias $\widehat{\theta}_2 \neq 0$. (1 marks)

UNSEEN

(iv) $\widehat{\theta}_2$ is better with respect to mean squared error since

$$
\begin{aligned}
& \frac{2\theta^2}{(n+1)(n+2)} \leq \frac{\theta^2}{3n} \\
\Leftrightarrow\ & 6n \leq (n+1)(n+2) \\
\Leftrightarrow\ & 6n \leq n^2 + 3n + 2 \\
\Leftrightarrow\ & 0 \leq n^2 - 3n + 2 \\
\Leftrightarrow\ & 0 \leq (n-1)(n-2).
\end{aligned}
$$

Both $\widehat{\theta}_1$ and $\widehat{\theta}_2$ have equal mean squared errors when $n = 1, 2$. (1 marks)

UNSEEN

Solutions to Question 3

ILOs addressed: define elementary statistical concepts and terminology such as confidence intervals and hypothesis tests.

(a) Suppose we wish to test $H_0 : \mu = \mu_0$ versus $H_0 : \mu \neq \mu_0$.

(i) the Type I error occurs if H_0 is rejected when in fact $\mu = \mu_0$; (1 marks)
SEEN

(ii) the Type II error occurs if H_0 is accepted when in fact $\mu \neq \mu_0$; (1 marks)
SEEN

(iii) the significance level is the probability of type I error. (1 marks)
SEEN

(b) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N(\mu, \sigma^2)$, where σ is known. The rejection region for the following tests are

(i) reject $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$ if $\sqrt{n} \left| \overline{X} - \mu_0 \right| / \sigma > z_{1-\frac{\alpha}{2}}$; (1 marks)
SEEN

(ii) reject $H_0 : \mu = \mu_0$ versus $H_1 : \mu < \mu_0$ if $\sqrt{n} \left(\overline{X} - \mu_0 \right) / \sigma < z_{\alpha}$. (1 marks)
SEEN

(c) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N(\mu, \sigma^2)$, where σ is not known. Then,

(i) the required probability is

$$
\begin{aligned}
\Pr(\text{Reject } H_0 \mid H_1 \text{ is true}) &= \Pr\left(\frac{\sqrt{n}\left|\overline{X}-\mu_0\right|}{\sigma} > z_{1-\frac{\alpha}{2}} \mid \mu \neq \mu_0\right) \\
&= \Pr\left(\frac{\sqrt{n}\left(\overline{X}-\mu_0\right)}{\sigma} > z_{1-\frac{\alpha}{2}} \text{ or } \frac{\sqrt{n}\left(\overline{X}-\mu_0\right)}{\sigma} < -z_{1-\frac{\alpha}{2}} \mid \mu \neq \mu_0\right) \\
&= \Pr\left(\frac{\sqrt{n}\left(\overline{X}-\mu_0\right)}{\sigma} > z_{1-\frac{\alpha}{2}} \mid \mu \neq \mu_0\right) + \Pr\Bigg(\frac{\sqrt{n}\left(\overline{X}-\mu_0\right)}{\sigma} < -z_{1-\frac{\alpha}{2}} \\
&= \Pr\left(\frac{\sqrt{n}\left(\overline{X}-\mu+\mu-\mu_0\right)}{\sigma} > z_{1-\frac{\alpha}{2}} \mid \mu \neq \mu_0\right) + \Pr\Bigg(\frac{\sqrt{n}\left(\overline{X}-\mu+\mu\right.}{\sigma} \\
&= \Pr\left(\frac{\sqrt{n}\left(\overline{X}-\mu\right)}{\sigma} > z_{1-\frac{\alpha}{2}} - \frac{\sqrt{n}\left(\mu-\mu_0\right)}{\sigma} \mid \mu \neq \mu_0\right) + \Pr\Bigg(\frac{\sqrt{n}\left(\overline{X}-\right.}{\sigma} \\
&= \Pr\left(N(0,1) > z_{1-\frac{\alpha}{2}} - \frac{\sqrt{n}\left(\mu-\mu_0\right)}{\sigma}\right) + \Pr\Bigg(N(0,1) < -z_{1-\frac{\alpha}{2}} - \frac{\sqrt{n}}{} \\
&= 1 - \Pr\left(N(0,1) \leq z_{1-\frac{\alpha}{2}} - \frac{\sqrt{n}\left(\mu-\mu_0\right)}{\sigma}\right) + \Pr\Bigg(N(0,1) < -z_{1-\frac{\alpha}{2}} - \\
&= 1 - \Phi\left(z_{1-\frac{\alpha}{2}} - \frac{\sqrt{n}\left(\mu-\mu_0\right)}{\sigma}\right) + \Phi\left(-z_{1-\frac{\alpha}{2}} - \frac{\sqrt{n}\left(\mu-\mu_0\right)}{\sigma}\right).
\end{aligned}
$$

(3 marks)

UNSEEN

(ii) the required probability is

$$
\begin{aligned}
\Pr(\text{Reject } H_0 \mid H_1 \text{ is true}) &= \Pr\left(\frac{\sqrt{n}\left(\overline{X}-\mu_0\right)}{\sigma} < z_\alpha \mid \mu < \mu_0\right) \\
&= \Pr\left(\frac{\sqrt{n}\left(\overline{X}-\mu+\mu-\mu_0\right)}{\sigma} < z_\alpha \mid \mu < \mu_0\right) \\
&= \Pr\left(\frac{\sqrt{n}\left(\overline{X}-\mu\right)}{\sigma} < z_\alpha - \frac{\sqrt{n}\left(\mu-\mu_0\right)}{\sigma} \mid \mu < \mu_0\right) \\
&= \Pr\left(N(0,1) < z_\alpha - \frac{\sqrt{n}\left(\mu-\mu_0\right)}{\sigma}\right) \\
&= \Phi\left(z_\alpha - \frac{\sqrt{n}\left(\mu-\mu_0\right)}{\sigma}\right).
\end{aligned}
$$

(2 marks)

UNSEEN

Solutions to Question 4

ILOs addressed: define elementary statistical concepts and terminology such as confidence intervals and hypothesis tests; conduct statistical inferences, including confidence intervals and hypothesis tests, in simple one and two-sample situations; sampling distributions.

(a) Let $\mathbf{X} = (X_1, \ldots, X_n)$, with $X_1, \ldots, X_n$ an independent random sample from a distribution F_X with unknown parameter θ. Let $I(\mathbf{X}) = [a(\mathbf{X}), b(\mathbf{X})]$ denote an interval estimator for θ.

(i) $I(\mathbf{X})$ is a $100(1-\alpha)\%$ confidence interval if

$$\Pr\left(a(\mathbf{X}) < \theta < b(\mathbf{X})\right) = 1 - \alpha;$$

(1 marks)

SEEN

(ii) the coverage probability of $I(\mathbf{X})$ is

$$\Pr\left(a(\mathbf{X}) < \theta < b(\mathbf{X})\right);$$

(1 marks)

SEEN

(iii) the coverage length of $I(\mathbf{X})$ is $b(\mathbf{X}) - a(\mathbf{X})$. (1 marks)

SEEN

(b) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N(\mu, \sigma^2)$.

(i) if σ is known then $\sqrt{n}\left(\overline{X} - \mu\right)/\sigma \sim N(0,1)$. So,

$$\begin{aligned}
& \Pr\left(z_{\alpha/2} < \frac{\sqrt{n}\left(\overline{X} - \mu\right)}{\sigma} < z_{1-\alpha/2}\right) = 1 - \alpha \\
\Leftrightarrow \ & \Pr\left(\frac{\sigma}{\sqrt{n}} z_{\alpha/2} < \overline{X} - \mu < \frac{\sigma}{\sqrt{n}} z_{1-\alpha/2}\right) = 1 - \alpha \\
\Leftrightarrow \ & \Pr\left(-\overline{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2} < -\mu < -\overline{X} + \frac{\sigma}{\sqrt{n}} z_{1-\alpha/2}\right) = 1 - \alpha \\
\Leftrightarrow \ & \Pr\left(\overline{X} - \frac{\sigma}{\sqrt{n}} z_{1-\alpha/2} < \mu < \overline{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right) = 1 - \alpha.
\end{aligned}$$

Hence, a $100(1-\alpha)\%$ confidence interval for μ is

$$\left(\overline{X} - \frac{\sigma}{\sqrt{n}} z_{1-\alpha/2}, \overline{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right).$$

(1 marks)

SEEN

(ii) if σ is not known then $\sqrt{n}\left(\overline{X}-\mu\right)/S \sim t_{n-1}$. So,

$$
\begin{aligned}
& \Pr\left(t_{n-1,\alpha/2} < \frac{\sqrt{n}\left(\overline{X}-\mu\right)}{S} < t_{n-1,1-\alpha/2}\right) = 1-\alpha \\
\Leftrightarrow\ & \Pr\left(\frac{S}{\sqrt{n}} t_{n-1,\alpha/2} < \overline{X}-\mu < \frac{S}{\sqrt{n}} t_{n-1,1-\alpha/2}\right) = 1-\alpha \\
\Leftrightarrow\ & \Pr\left(-\overline{X}+\frac{S}{\sqrt{n}} t_{n-1,\alpha/2} < -\mu < -\overline{X}+\frac{S}{\sqrt{n}} t_{n-1,1-\alpha/2}\right) = 1-\alpha \\
\Leftrightarrow\ & \Pr\left(\overline{X}-\frac{S}{\sqrt{n}} t_{n-1,1-\alpha/2} < \mu < \overline{X}-\frac{S}{\sqrt{n}} t_{n-1,\alpha/2}\right) = 1-\alpha.
\end{aligned}
$$

Hence, a $100(1-\alpha)\%$ confidence interval for μ is

$$
\left(\overline{X}-\frac{S}{\sqrt{n}} t_{n-1,1-\alpha/2}, \overline{X}-\frac{S}{\sqrt{n}} t_{n-1,\alpha/2}\right).
$$

(1 marks)

SEEN

(c) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $Uniform\,[0,a]$.

(i) The cumulative distribution function $\max\left(X_1, X_2, \ldots, X_n\right) = Z$ say, is

$$
\begin{aligned}
F_Z(z) &= \Pr\left(Z \leq z\right) \\
&= \Pr\left(\max\left(X_1, X_2, \ldots, X_n\right) \leq z\right) \\
&= \Pr\left(X_1 \leq z, X_2 \leq z, \ldots, X_n \leq z\right) \\
&= \Pr\left(X_1 \leq z\right)\Pr\left(X_2 \leq z\right)\cdots\Pr\left(X_n \leq z\right) \\
&= \left(\frac{z}{a}\right)\left(\frac{z}{a}\right)\cdots\left(\frac{z}{a}\right) \\
&= \left(\frac{z}{a}\right)^n
\end{aligned}
$$

for $0 < z < a$. (2 marks)

UNSEEN

(ii) The $\left(\frac{\alpha}{2}\right)$th and $\left(1-\frac{\alpha}{2}\right)$th percentiles of Z are $a\left(\frac{\alpha}{2}\right)^{1/n}$ and $a\left(1-\frac{\alpha}{2}\right)^{1/n}$, respectively. So,

$$
\Pr\left(a\left(\frac{\alpha}{2}\right)^{1/n} \leq Z \leq a\left(1-\frac{\alpha}{2}\right)^{1/n}\right) = 1-\alpha,
$$

which can be rewritten as

$$
\Pr\left(Z\left(1-\frac{\alpha}{2}\right)^{-1/n} \leq a \leq Z\left(\frac{\alpha}{2}\right)^{-1/n}\right) = 1-\alpha.
$$

Hence, a $100(1-\alpha)\%$ confidence interval for a is

$$\left[Z\left(1-\frac{\alpha}{2}\right)^{-1/n}, Z\left(\frac{\alpha}{2}\right)^{-1/n} \right].$$

(3 marks)

UNSEEN

Solutions to Question 5

ILOs addressed: analyse and compare statistical properties of simple estimators.

Suppose X_1 and X_2 are independent Exp$(1/\lambda)$ random variables. Let $\widehat{\theta_1} = a\left(X_1 + X_2\right)$ and $\widehat{\theta_2} = b\sqrt{X_1 X_2}$ denote possible estimators of λ, where a and b are constants.

(i) The expectation of $\widehat{\theta_1}$ is

$$
\begin{aligned}
E\left(\widehat{\theta_1}\right) &= a\left[E\left(X_1\right) + E\left(X_2\right)\right] \\
&= 2a \int_0^{+\infty} \frac{x}{\lambda} \exp\left(-\frac{x}{\lambda}\right) dx \\
&= 2a\lambda \int_0^{+\infty} y \exp\left(-y\right) dy \\
&= 2a\lambda.
\end{aligned}
$$

So, $\widehat{\theta_1}$ is unbiased for λ if $a = 1/2$. (4 marks)

UNSEEN

(ii) The expectation of $\widehat{\theta_2}$ is

$$
\begin{aligned}
E\left(\widehat{\theta_2}\right) &= bE\left(\sqrt{X_1}\right) E\left(\sqrt{X_2}\right) \\
&= b\left[\int_0^{+\infty} \frac{\sqrt{x}}{\lambda} \exp\left(-\frac{x}{\lambda}\right) dx\right]^2 \\
&= b\left[\int_0^{+\infty} \sqrt{\lambda y} \exp\left(-y\right) dy\right]^2 \\
&= b\lambda\left[\int_0^{+\infty} \sqrt{y} \exp\left(-y\right) dy\right]^2 \\
&= b\lambda\left[\Gamma(3/2)\right]^2 \\
&= b\lambda\pi/4.
\end{aligned}
$$

So, $\widehat{\theta_2}$ is unbiased for λ if $b = 4/\pi$. (4 marks)

UNSEEN

(iii) The variance of $\widehat{\theta_1}$ is

$$
\begin{aligned}
\text{Var}\left(\widehat{\theta_1}\right) &= a^2 \left[\text{Var}\left(X_1\right) + \text{Var}\left(X_2\right)\right] \\
&= \\
&= 2a^2 \text{Var}\left(X_1\right) \\
&= 2a^2 \left[\int_0^{+\infty} \frac{x^2}{\lambda} \exp\left(-\frac{x}{\lambda}\right) dx - \lambda^2\right] \\
&= 2a^2 \left[\lambda^2 \int_0^{+\infty} y^2 \exp\left(-y\right) dy - \lambda^2\right] \\
&= 2a^2 \left[\lambda^2 \Gamma(3) - \lambda^2\right] \\
&= 2a^2 \lambda^2 \\
&= \lambda^2/2.
\end{aligned}
$$

(4 marks)

UNSEEN

(iv) The variance of $\widehat{\theta_2}$ is

$$
\text{Var}\left(\widehat{\theta_2}\right) = b^2 E\left(X_1\right) E\left(X_2\right) - \lambda^2 = b^2 \lambda^2 - \lambda^2 = \left(b^2 - 1\right) \lambda^2 = \left(\frac{16}{\pi^2} - 1\right) \lambda^2.
$$

(4 marks)

UNSEEN

(v) Clearly, $\lambda^2/2 < \left(\frac{16}{\pi^2} - 1\right) \lambda^2$, so the estimator $\widehat{\theta_1}$ is better with respect to mean squared error. (4 marks)

UNSEEN

Solutions to Question 6

ILOs addressed: analyse statistical properties of simple estimators.

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution specified by the probability density function $\frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right)$ for $x > 0$.

(i) The likelihood function of σ^2 is

$$\begin{aligned} L\left(\sigma^2\right) &= \prod_{i=1}^{n} \left[\frac{X_i}{\sigma^2} \exp\left(-\frac{X_i^2}{2\sigma^2}\right)\right] \\ &= \frac{1}{\sigma^{2n}} \left(\prod_{i=1}^{n} X_i\right) \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} X_i^2\right). \end{aligned}$$

(4 marks)

UNSEEN

(ii) The log likelihood function of σ^2 is

$$\log L\left(\sigma^2\right) = -2n \log \sigma + \prod_{i=1}^{n} \log X_i - \frac{1}{2\sigma^2} \sum_{i=1}^{n} X_i^2.$$

The derivative with respect to σ is

$$\frac{d \log L\left(\sigma^2\right)}{d\sigma} = -\frac{2n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} X_i^2.$$

Setting this to zero gives

$$\widehat{\sigma^2} = \frac{1}{2n} \sum_{i=1}^{n} X_i^2.$$

This is a maximum likelihood estimator since

$$\begin{aligned} \frac{d^2 \log L\left(\sigma^2\right)}{d\sigma^2} &= \frac{2n}{\sigma^2} - \frac{3}{\sigma^4} \sum_{i=1}^{n} X_i^2 \\ &= \frac{1}{\sigma^4} \left[2n\sigma^2 - 3 \sum_{i=1}^{n} X_i^2\right] \\ &= \frac{1}{\sigma^4} \left[2n \frac{1}{2n} \sum_{i=1}^{n} X_i^2 - 3 \sum_{i=1}^{n} X_i^2\right] \\ &< 0 \end{aligned}$$

at $\sigma = \widehat{\sigma}$. (4 marks)

UNSEEN

(iii) By the invariance principle, the maximum likelihood estimator of σ is

$$\widehat{\sigma} = \sqrt{\frac{1}{2n}\sum_{i=1}^{n} X_i^2}.$$

(4 marks)

UNSEEN

(iv) The bias of $\widehat{\sigma^2}$ is

$$\begin{aligned}
\text{Bias}\left(\widehat{\sigma^2}\right) &= E\left(\widehat{\sigma^2}\right) - \sigma^2 \\
&= E\left(\frac{1}{2n}\sum_{i=1}^{n} X_i^2\right) - \sigma^2 \\
&= \frac{1}{2n}\sum_{i=1}^{n} E\left(X_i^2\right) - \sigma^2 \\
&= \frac{1}{2n\sigma^2}\sum_{i=1}^{n}\int_0^\infty x^3 \exp\left(-\frac{x^2}{2\sigma^2}\right) dx - \sigma^2 \\
&= \frac{\sigma^2}{n}\sum_{i=1}^{n}\int_0^\infty y \exp(-y)\, dy - \sigma^2 \\
&= \frac{\sigma^2}{n}\sum_{i=1}^{n}\Gamma(2) - \sigma^2 \\
&= \frac{\sigma^2}{n}\sum_{i=1}^{n} 1 - \sigma^2 \\
&= 0.
\end{aligned}$$

Hence, $\widehat{\sigma^2}$ is unbiased for σ^2. (4 marks)

UNSEEN

(v) The mean squared error of $\widehat{\sigma^2}$ is

$$
\begin{aligned}
\text{MSE}\left(\widehat{\sigma^2}\right) &= \text{Var}\left(\widehat{\sigma^2}\right) \\
&= \text{Var}\left(\frac{1}{2n}\sum_{i=1}^{n} X_i^2\right) \\
&= \frac{1}{4n^2}\sum_{i=1}^{n} \text{Var}\left(X_i^2\right) \\
&= \frac{1}{4n^2}\sum_{i=1}^{n} \left\{E\left(X_i^4\right) - \left[E\left(X_i^2\right)\right]^2\right\} \\
&= \frac{1}{4n^2}\sum_{i=1}^{n} \left\{E\left(X_i^4\right) - \left[2\sigma^2\right]^2\right\} \\
&= \frac{1}{4n^2}\sum_{i=1}^{n} \left\{4\sigma^4 \int_0^\infty y^2 \exp(-y)\, dy - 4\sigma^4\right\} \\
&= \frac{1}{4n^2}\sum_{i=1}^{n} \left\{4\sigma^4 \Gamma(3) - 4\sigma^4\right\} \\
&= \frac{1}{4n^2}\sum_{i=1}^{n} \left\{8\sigma^4 - 4\sigma^4\right\} \\
&= \frac{\sigma^2}{n}.
\end{aligned}
$$

Hence, $\widehat{\sigma^2}$ is consistent σ^2. (4 marks)

UNSEEN

Solutions to Question 7

ILOs addressed: analyse statistical properties of simple estimators.

An electrical circuit consists of four batteries connected in series to a lightbulb. We model the battery lifetimes X_1, X_2, X_3, X_4 as independent and identically distributed $Uni(0, \theta)$ random variables. Our experiment to measure the operating time of the circuit is stopped when any one of the batteries fails. Hence, the only random variable we observe is $Y = \min(X_1, X_2, X_3, X_4)$.

(i) The cdf of Y is

$$\begin{aligned} \Pr(Y \leq y) &= 1 - \Pr\left[\min\left(X_1, X_2, X_3, X_4\right) > y\right] \\ &= 1 - \Pr\left(X_1 > y\right) \Pr\left(X_2 > y\right) \Pr\left(X_3 > y\right) \Pr\left(X_4 > y\right) \\ &= 1 - \Pr^4\left(X > y\right) \\ &= 1 - (1 - y/\theta)^4. \end{aligned}$$

(4 marks)

UNSEEN

(ii) The likelihood function of θ is

$$L(\theta) = 4(\theta - y)^3/\theta^4$$

for $0 < y < \theta$. (4 marks)

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(iii) The log-likelihood function is

$$\log L(\theta) = \log 4 + 3\log(\theta - y)^3 - 4\log\theta$$

and

$$\frac{d \log L(\theta)}{d\theta} = \frac{3}{\theta - y} - \frac{4}{\theta}.$$

Setting $d\log L(\theta)/d\theta = 0$ gives $\widehat{\theta} = 4y$. This is an MLE since

$$\frac{d^2 \log L(\theta)}{d\theta^2} = -\frac{3}{(\theta - y)^2} + \frac{4}{\theta^2}$$

at $\widehat{\theta} = 4y$ is negative. (4 marks)

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(iv) The bias of $\widehat{\theta}$ is

$$
\begin{aligned}
Bias\left(\widehat{\theta}\right) &= E\left(\widehat{\theta}\right) - \theta \\
&= 16\theta^{-4}\int_0^\theta y(\theta-y)^3 dy - \theta \\
&= 16\theta\int_0^1 y(1-y)^3 dy - \theta \\
&= \frac{4\theta}{5} - \theta \\
&= -\frac{\theta}{5},
\end{aligned}
$$

so the estimator is biased. (4 marks)

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(v) The variance of $\widehat{\theta}$ is

$$
\begin{aligned}
Var\left(\widehat{\theta}\right) &= E\left(\widehat{\theta}^2\right) - E^2\left(\widehat{\theta}\right) \\
&= 64\theta^{-4}\int_0^\theta y^2(\theta-y)^3 dy - \frac{16\theta^2}{25} \\
&= 64\theta^2\int_0^1 y^2(1-y)^3 dy - \frac{16\theta^2}{25} \\
&= \frac{16\theta^2}{15} - \frac{16\theta^2}{25} \\
&= \frac{32\theta^2}{75}.
\end{aligned}
$$

So, the mean squared error of $\widehat{\lambda}$ is

$$
MSE\left(\widehat{\theta}\right) = Var\left(\widehat{\theta}\right) + Bias^2\left(\widehat{\theta}\right) = \frac{32\theta^2}{75} + \frac{\theta^2}{25} = \frac{35\theta^2}{75} = \frac{7\theta^2}{15}.
$$

(4 marks)

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