

**SOLUTIONS TO
MATH48181
EXTREME VALUES
AND FINANCIAL RISK EXAM**

Solutions to Question A1

a) The joint cdf of X and Y is

$$\begin{aligned} F_{X,Y}(x, y) &= \int_0^y \int_0^x \frac{4uv + 2u + 2v + 1}{4} dudv \\ &= \frac{1}{4} \int_0^y [2u^2v + u^2 + 2uv + u]_0^x dv \\ &= \frac{1}{4} \int_0^y [2x^2v + x^2 + 2xv + x] dv \\ &= \frac{1}{4} [x^2v^2 + x^2v + xv^2 + xv]_0^y \\ &= \frac{1}{4} [x^2y^2 + x^2y + xy^2 + xy]. \end{aligned}$$

(3 marks)

UNSEEN

b) The marginal cdfs of X and Y are

$$F_X(x) = F_{X,Y}(x, 1) = \frac{1}{4} [x^2 + x^2 + x + x] = \frac{x(x+1)}{2}$$

and

$$F_Y(y) = F_{X,Y}(1, y) = \frac{1}{4} [y^2 + y + y^2 + y] = \frac{y(y+1)}{2}.$$

(2 marks)

UNSEEN

c) First note that $w(F_X) = 1$. F_X belongs to the Weibull max domain of attraction since

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{1 - F_X(1 - tx)}{1 - F_X(1 - t)} &= \lim_{t \rightarrow 0} \frac{1 - \frac{(1-tx)(2-tx)}{2}}{1 - \frac{(1-t)(2-t)}{2}} \\ &= \lim_{t \rightarrow 0} \frac{2 - (1-tx)(2-tx)}{2 - (1-t)(2-t)} \\ &= \lim_{t \rightarrow 0} \frac{3tx - t^2x^2}{3t - t^2} \\ &= \lim_{t \rightarrow 0} \frac{3x - tx^2}{3 - t} \\ &= x. \end{aligned}$$

(2 marks)

UNSEEN

d) First note that $w(F_Y) = 1$. F_Y belongs to the Weibull max domain of attraction since

$$\begin{aligned}\lim_{t \rightarrow 0} \frac{1 - F_Y(1 - ty)}{1 - F_Y(1 - t)} &= \lim_{t \rightarrow 0} \frac{1 - \frac{(1-ty)(2-ty)}{2}}{1 - \frac{(1-t)(2-t)}{2}} \\ &= \lim_{t \rightarrow 0} \frac{2 - (1-ty)(2-ty)}{2 - (1-t)(2-t)} \\ &= \lim_{t \rightarrow 0} \frac{3ty - t^2y^2}{3t - t^2} \\ &= \lim_{t \rightarrow 0} \frac{3y - ty^2}{3 - t} \\ &= y.\end{aligned}$$

(2 marks)

UNSEEN

e) Use the formulas $a_n = w(F_X) - F_X^{-1}\left(1 - \frac{1}{n}\right)$ and $b_n = 1$. Inverting

$$F_X(x) = \frac{x(x+1)}{2} = p$$

gives

$$x^2 + x - 2p = 0,$$

which has the valid root

$$x = \frac{-1 + \sqrt{1 + 8p}}{2}.$$

So,

$$F_X^{-1}\left(1 - \frac{1}{n}\right) = \frac{-1 + \sqrt{1 + 8\left(1 - \frac{1}{n}\right)}}{2} = \frac{-1 + 3\sqrt{1 - \frac{8}{9n}}}{2}.$$

Hence,

$$a_n = 1 - \frac{-1 + 3\sqrt{1 - \frac{8}{9n}}}{2}$$

and $b_n = 1$.

(2 marks)

UNSEEN

f) Use the formulas $c_n = w(F_Y) - F_Y^{-1}\left(1 - \frac{1}{n}\right)$ and $d_n = 1$. Inverting

$$F_Y(y) = \frac{y(y+1)}{2} = p$$

gives

$$y^2 + y - 2p = 0,$$

which has the valid root

$$y = \frac{-1 + \sqrt{1 + 8p}}{2}.$$

So,

$$F_Y^{-1}\left(1 - \frac{1}{n}\right) = \frac{-1 + \sqrt{1 + 8\left(1 - \frac{1}{n}\right)}}{2} = \frac{-1 + 3\sqrt{1 - \frac{8}{9n}}}{2}.$$

Hence,

$$c_n = 1 - \frac{-1 + 3\sqrt{1 - \frac{8}{9n}}}{2}$$

and $d_n = 1$.

(2 marks)

UNSEEN

g) The limiting cdf is

$$\begin{aligned} & \lim_{n \rightarrow \infty} F_{X,Y}(a_n x + b_n, c_n y + d_n) \\ &= \lim_{n \rightarrow \infty} \frac{1}{4^n} \left[(a_n x + 1)^2 (a_n y + 1)^2 + (a_n x + 1)^2 (a_n y + 1) + (a_n x + 1) (a_n y + 1)^2 + (a_n x + 1) (a_n y + 1) \right]^n \\ &= \lim_{n \rightarrow \infty} \frac{1}{4^n} (a_n x + 1)^{2n} (a_n y + 1)^{2n} \left[1 + (a_n y + 1)^{-1} + (a_n x + 1)^{-1} + (a_n x + 1)^{-1} (a_n y + 1)^{-1} \right]^n. \end{aligned}$$

Now note that

$$a_n = 1 - \frac{-1 + 3\sqrt{1 - \frac{8}{9n}}}{2} = 1 - \frac{-1 + 3\left(1 - \frac{4}{9n} + \dots\right)}{2} = \frac{2}{3n} + \dots$$

as $n \rightarrow \infty$. Hence,

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} F_{X,Y}(a_n x + b_n, c_n y + d_n) \\
 = & \lim_{n \rightarrow \infty} \frac{1}{4^n} \left(\frac{2}{3n} x + 1 \right)^{2n} \left(\frac{2}{3n} y + 1 \right)^{2n} [1 + (a_n y + 1)^{-1} + (a_n x + 1)^{-1} + (a_n x + 1)^{-1} (a_n y + 1)^{-1}]^n \\
 = & \lim_{n \rightarrow \infty} \frac{1}{4^n} \exp\left(\frac{4x}{3}\right) \exp\left(\frac{4y}{3}\right) [1 + (0 + 1)^{-1} + (0 + 1)^{-1} + (0 + 1)^{-1} (0 + 1)^{-1}]^n \\
 = & \lim_{n \rightarrow \infty} \frac{1}{4^n} \exp\left(\frac{4x}{3}\right) \exp\left(\frac{4y}{3}\right) 4^n \\
 = & \exp\left(\frac{4x}{3} + \frac{4y}{3}\right).
 \end{aligned}$$

(5 marks)

UNSEEN

h) Yes, the extremes are completely independent since

$$\begin{aligned}
 \lim_{n \rightarrow \infty} F_{X,Y}(a_n x + b_n, c_n y + d_n) &= \exp\left(\frac{4x}{3} + \frac{4y}{3}\right) \\
 &= \exp\left(\frac{4x}{3}\right) \exp\left(\frac{4y}{3}\right) \\
 &= \lim_{n \rightarrow \infty} F_X(a_n x + b_n) \lim_{n \rightarrow \infty} F_Y(c_n y + d_n).
 \end{aligned}$$

(2 marks)

UNSEEN

Solutions to Question A2

$C(u_1, u_2)$ is a valid copula if

$$C(u, 0) = 0,$$

$$C(0, u) = 0,$$

$$C(1, u) = u,$$

$$C(u, 1) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) \geq 0.$$

(4 marks)

SEEN

a) for the copula defined by $C(u_1, u_2) = \min(u_1^{1-\alpha} u_2, u_1 u_2^{1-\beta})$, we have

$$C(u, 0) = \min(0, 0) = 0,$$

$$C(0, u) = \min(0, 0) = 0,$$

$$C(1, u) = \min(u, u^{1-\beta}) = u,$$

$$C(u, 1) = \min(u^{1-\alpha}, u) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \frac{\partial}{\partial u_1} \begin{cases} u_1^{1-\alpha} u_2, & \text{if } u_1^{-\alpha} \leq u_2^{-\beta}, \\ u_1 u_2^{1-\beta}, & \text{if } u_1^{-\alpha} > u_2^{-\beta} \end{cases} = \begin{cases} (1-\alpha) u_1^{-\alpha} u_2, & \text{if } u_1^{-\alpha} \leq u_2^{-\beta}, \\ u_2^{1-\beta}, & \text{if } u_1^{-\alpha} > u_2^{-\beta} \end{cases} \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \frac{\partial}{\partial u_2} \begin{cases} u_1^{1-\alpha} u_2, & \text{if } u_1^{-\alpha} \leq u_2^{-\beta}, \\ u_1 u_2^{1-\beta}, & \text{if } u_1^{-\alpha} > u_2^{-\beta} \end{cases} = \begin{cases} u_1^{1-\alpha}, & \text{if } u_1^{-\alpha} \leq u_2^{-\beta}, \\ (1-\beta) u_1 u_2^{-\beta}, & \text{if } u_1^{-\alpha} > u_2^{-\beta} \end{cases} \geq 0.$$

(4 marks)

UNSEEN

b) for the copula defined by $C(u_1, u_2) = u_2 - \left[\max \left((1 - u_1)^{1/n} + u_2^{1/n} - 1, 0 \right) \right]^n$, we have

$$C(u, 0) = 0 - \left[\max \left((1 - u)^{1/n} + 0 - 1, 0 \right) \right]^n = 0,$$

$$C(0, u) = u - \left[\max \left((1 - 0)^{1/n} + u^{1/n} - 1, 0 \right) \right]^n = u - u = 0,$$

$$C(1, u) = u - \left[\max \left((1 - 1)^{1/n} + u^{1/n} - 1, 0 \right) \right]^n = u - 0 = u,$$

$$C(u, 1) = 1 - \left[\max \left((1 - u)^{1/n} + 1 - 1, 0 \right) \right]^n = 1 - (1 - u) = u,$$

$$\begin{aligned} \frac{\partial}{\partial u_1} C(u_1, u_2) &= \frac{\partial}{\partial u_1} \begin{cases} u_2 - \left[(1 - u_1)^{1/n} + u_2^{1/n} - 1 \right]^n, & \text{if } (1 - u_1)^{1/n} + u_2^{1/n} \geq 1, \\ u_2, & \text{if } (1 - u_1)^{1/n} + u_2^{1/n} < 1 \end{cases} \\ &= \begin{cases} \left[(1 - u_1)^{1/n} + u_2^{1/n} - 1 \right]^{n-1} (1 - u_1)^{1/n-1}, & \text{if } (1 - u_1)^{1/n} + u_2^{1/n} \geq 1, \\ 0, & \text{if } (1 - u_1)^{1/n} + u_2^{1/n} < 1 \end{cases} \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial u_2} C(u_1, u_2) &= \frac{\partial}{\partial u_2} \begin{cases} u_2 - \left[(1 - u_1)^{1/n} + u_2^{1/n} - 1 \right]^n, & \text{if } (1 - u_1)^{1/n} + u_2^{1/n} \geq 1, \\ u_2, & \text{if } (1 - u_1)^{1/n} + u_2^{1/n} < 1 \end{cases} \\ &= \begin{cases} 1 - \left[(1 - u_1)^{1/n} + u_2^{1/n} - 1 \right]^{n-1} u_2^{1/n-1}, & \text{if } (1 - u_1)^{1/n} + u_2^{1/n} \geq 1, \\ 1, & \text{if } (1 - u_1)^{1/n} + u_2^{1/n} < 1 \end{cases} \\ &= \begin{cases} 1 - \left[u_2^{-1/n} \left((1 - u_1)^{1/n} + u_2^{1/n} - 1 \right) \right]^{n-1}, & \text{if } (1 - u_1)^{1/n} + u_2^{1/n} \geq 1, \\ 1, & \text{if } (1 - u_1)^{1/n} + u_2^{1/n} < 1 \end{cases} \\ &\geq 0. \end{aligned}$$

(4 marks)

UNSEEN

c) for the copula defined by $C(u_1, u_2) = \log_\alpha \left[1 + \frac{(\alpha^{u_1} - 1)(\alpha^{u_2} - 1)}{\alpha - 1} \right]$, we have

$$C(u, 0) = \log_\alpha \left[1 + \frac{(\alpha^u - 1)(1 - 1)}{\alpha - 1} \right] = 0,$$

$$C(0, u) = \log_{\alpha} \left[1 + \frac{(1-1)(\alpha^u - 1)}{\alpha - 1} \right] = 0,$$

$$C(1, u) = \log_{\alpha} \left[1 + \frac{(\alpha - 1)(\alpha^u - 1)}{\alpha - 1} \right] = u \log_{\alpha} \alpha = u,$$

$$C(u, 1) = \log_{\alpha} \left[1 + \frac{(\alpha^u - 1)(\alpha - 1)}{\alpha - 1} \right] = u \log_{\alpha} \alpha = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \left[1 + \frac{(\alpha^{u_1} - 1)(\alpha^{u_2} - 1)}{\alpha - 1} \right]^{-1} \frac{\alpha^{u_1}(\alpha^{u_2} - 1)}{\alpha - 1} \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \left[1 + \frac{(\alpha^{u_1} - 1)(\alpha^{u_2} - 1)}{\alpha - 1} \right]^{-1} \frac{\alpha^{u_2}(\alpha^{u_1} - 1)}{\alpha - 1} \geq 0.$$

(4 marks)

UNSEEN

d) for the copula defined by $\left\{ \left[(u_1^{-\theta} - 1)^{\delta} + (u_2^{-\theta} - 1)^{\delta} \right]^{1/\delta} + 1 \right\}^{-1/\theta}$, we have

$$C(u, 0) = \left\{ \left[(u^{-\theta} - 1)^{\delta} + (0^{-\theta} - 1)^{\delta} \right]^{1/\delta} + 1 \right\}^{-1/\theta} = 0,$$

$$C(0, u) = \left\{ \left[(0^{-\theta} - 1)^{\delta} + (u^{-\theta} - 1)^{\delta} \right]^{1/\delta} + 1 \right\}^{-1/\theta} = 0,$$

$$C(1, u) = \left\{ \left[(1 - 1)^{\delta} + (u^{-\theta} - 1)^{\delta} \right]^{1/\delta} + 1 \right\}^{-1/\theta} = u,$$

$$C(u, 1) = \left\{ \left[(u^{-\theta} - 1)^{\delta} + (1 - 1)^{\delta} \right]^{1/\delta} + 1 \right\}^{-1/\theta} = u,$$

$$\begin{aligned} \frac{\partial}{\partial u_1} C(u_1, u_2) &= \left\{ \left[(u_1^{-\theta} - 1)^{\delta} + (u_2^{-\theta} - 1)^{\delta} \right]^{1/\delta} + 1 \right\}^{-1/\theta-1} \\ &\quad \cdot \left[(u_1^{-\theta} - 1)^{\delta} + (u_2^{-\theta} - 1)^{\delta} \right]^{1/\delta-1} (u_1^{-\theta} - 1)^{\delta-1} u_1^{-\theta-1} \geq 0 \end{aligned}$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \left\{ \left[(u_1^{-\theta} - 1)^\delta + (u_2^{-\theta} - 1)^\delta \right]^{1/\delta} + 1 \right\}^{-1/\theta-1} \\ \cdot \left[(u_1^{-\theta} - 1)^\delta + (u_2^{-\theta} - 1)^\delta \right]^{1/\delta-1} (u_2^{-\theta} - 1)^{\delta-1} u_2^{-\theta-1} \geq 0.$$

(4 marks)

UNSEEN

Solutions to Question A3

a) We can write

$$\bar{G}(x, y) = \exp \left[-(x + y)A \left(\frac{y}{x + y} \right) \right]$$

for $x > 0$ and $y > 0$, where

$$A(w) = \begin{cases} A_1(w), & \text{if } 0 \leq w \leq w_1, \\ A_2(w), & \text{if } w_1 < w \leq w_2, \\ \vdots & \\ A_p(w), & \text{if } w_{p-1} \leq w \leq 1. \end{cases}$$

We now check the conditions for $A(\cdot)$. Clearly,

$$A(0) = A_1(0) = 1$$

and

$$A(1) = A_p(1) = 1.$$

Also $A(t) \geq 0$ since $A_i(w) \geq 0$ for all w and every k .

Also since each $A_i(w) \leq 1$,

$$A(w) \leq 1.$$

Also since each $A_i(w) \geq \max(w, 1 - w)$,

$$A(w) \geq \max(w, 1 - w).$$

Also since each $A_i(w)$ is convex,

$$A''(w) = \begin{cases} A_1''(w), & \text{if } 0 \leq w \leq w_1, \\ A_2''(w), & \text{if } w_1 < w \leq w_2, \\ \vdots & \\ A_p''(w), & \text{if } w_{p-1} \leq w \leq 1 \end{cases} \geq 0.$$

(7 marks)

UNSEEN

b) Note that

$$\bar{G}(x, 0) = \exp \{ -(x + 0)A_1(0) \} = \exp(-x)$$

and

$$\overline{G}(0, y) = \exp \{-(0 + y)A_p(1)\} = \exp(-y).$$

So, the joint cdf is

$$G(x, y) = \begin{cases} 1 - \exp(-x) - \exp(-y) + \exp \left\{ -(x + y)A_1 \left(\frac{y}{x+y} \right) \right\}, & \text{if } 0 \leq \frac{y}{x+y} \leq w_1, \\ 1 - \exp(-x) - \exp(-y) + \exp \left\{ -(x + y)A_2 \left(\frac{y}{x+y} \right) \right\}, & \text{if } w_1 < \frac{y}{x+y} \leq w_2, \\ \vdots \\ 1 - \exp(-x) - \exp(-y) + \exp \left\{ -(x + y)A_p \left(\frac{y}{x+y} \right) \right\}, & \text{if } w_{p-1} \leq \frac{y}{x+y} \leq 1. \end{cases}$$

(1 marks)

UNSEEN

c) the derivative of joint cdf with respect to x is

$$\frac{\partial G(x, y)}{\partial x} = \begin{cases} \exp(-x) + \left[\frac{y}{x+y}A'_1 \left(\frac{y}{x+y} \right) - A_1 \left(\frac{y}{x+y} \right) \right] \exp \left\{ -(x + y)A_1 \left(\frac{y}{x+y} \right) \right\}, & \text{if } 0 \leq \frac{y}{x+y} \leq w_1, \\ \exp(-x) + \left[\frac{y}{x+y}A'_2 \left(\frac{y}{x+y} \right) - A_2 \left(\frac{y}{x+y} \right) \right] \exp \left\{ -(x + y)A_2 \left(\frac{y}{x+y} \right) \right\}, & \text{if } w_1 < \frac{y}{x+y} \leq w_2, \\ \vdots \\ \exp(-x) + \left[\frac{y}{x+y}A'_p \left(\frac{y}{x+y} \right) - A_p \left(\frac{y}{x+y} \right) \right] \exp \left\{ -(x + y)A_p \left(\frac{y}{x+y} \right) \right\}, & \text{if } w_{p-1} \leq \frac{y}{x+y} \leq 1, \end{cases}$$

so the conditional cdf of Y given $X = x$ is

$$G(y | x) = \begin{cases} 1 + \left[\frac{y}{x+y}A'_1 \left(\frac{y}{x+y} \right) - A_1 \left(\frac{y}{x+y} \right) \right] \exp \left\{ x - (x + y)A_1 \left(\frac{y}{x+y} \right) \right\}, & \text{if } 0 \leq \frac{y}{x+y} \leq w_1, \\ 1 + \left[\frac{y}{x+y}A'_2 \left(\frac{y}{x+y} \right) - A_2 \left(\frac{y}{x+y} \right) \right] \exp \left\{ x - (x + y)A_2 \left(\frac{y}{x+y} \right) \right\}, & \text{if } w_1 < \frac{y}{x+y} \leq w_2, \\ \vdots \\ 1 + \left[\frac{y}{x+y}A'_p \left(\frac{y}{x+y} \right) - A_p \left(\frac{y}{x+y} \right) \right] \exp \left\{ x - (x + y)A_p \left(\frac{y}{x+y} \right) \right\}, & \text{if } w_{p-1} \leq \frac{y}{x+y} \leq 1. \end{cases}$$

(4 marks)

UNSEEN

d) the derivative of joint cdf with respect to y is

$$\frac{\partial G(x, y)}{\partial y} = \begin{cases} \exp(-y) + \left[\frac{x}{x+y}A'_1 \left(\frac{y}{x+y} \right) + A_1 \left(\frac{y}{x+y} \right) \right] \exp \left\{ -(x + y)A_1 \left(\frac{y}{x+y} \right) \right\}, & \text{if } 0 \leq \frac{y}{x+y} \leq w_1, \\ \exp(-y) + \left[\frac{x}{x+y}A'_2 \left(\frac{y}{x+y} \right) + A_2 \left(\frac{y}{x+y} \right) \right] \exp \left\{ -(x + y)A_2 \left(\frac{y}{x+y} \right) \right\}, & \text{if } w_1 < \frac{y}{x+y} \leq w_2, \\ \vdots \\ \exp(-y) + \left[\frac{x}{x+y}A'_p \left(\frac{y}{x+y} \right) + A_p \left(\frac{y}{x+y} \right) \right] \exp \left\{ -(x + y)A_p \left(\frac{y}{x+y} \right) \right\}, & \text{if } w_{p-1} \leq \frac{y}{x+y} \leq 1, \end{cases}$$

so the conditional cdf of X given $Y = y$ is

$$G(x | y) = \begin{cases} 1 + \left[\frac{x}{x+y} A'_1 \left(\frac{y}{x+y} \right) + A_1 \left(\frac{y}{x+y} \right) \right] \exp \left\{ y - (x+y) A_1 \left(\frac{y}{x+y} \right) \right\}, & \text{if } 0 \leq \frac{y}{x+y} \leq w_1, \\ 1 + \left[\frac{x}{x+y} A'_2 \left(\frac{y}{x+y} \right) + A_2 \left(\frac{y}{x+y} \right) \right] \exp \left\{ y - (x+y) A_2 \left(\frac{y}{x+y} \right) \right\}, & \text{if } w_1 < \frac{y}{x+y} \leq w_2, \\ \vdots \\ 1 + \left[\frac{x}{x+y} A'_p \left(\frac{y}{x+y} \right) + A_p \left(\frac{y}{x+y} \right) \right] \exp \left\{ y - (x+y) A_p \left(\frac{y}{x+y} \right) \right\}, & \text{if } w_{p-1} \leq \frac{y}{x+y} \leq 1. \end{cases}$$

(4 marks)

UNSEEN

e) the derivative of joint cdf with respect to x and y is

$$g(x, y) = \begin{cases} \left\{ \left[\frac{y}{x+y} A'_1 \left(\frac{y}{x+y} \right) - A_1 \left(\frac{y}{x+y} \right) \right] \left[\frac{x}{x+y} A'_1 \left(\frac{y}{x+y} \right) + A_1 \left(\frac{y}{x+y} \right) \right] + \frac{xy}{(x+y)^2} A''_1 \left(\frac{y}{x+y} \right) \right\} \\ \cdot \exp \left\{ -(x+y) A_1 \left(\frac{y}{x+y} \right) \right\}, & \text{if } 0 \leq \frac{y}{x+y} \leq w_1, \\ \left\{ \left[\frac{y}{x+y} A'_2 \left(\frac{y}{x+y} \right) - A_2 \left(\frac{y}{x+y} \right) \right] \left[\frac{x}{x+y} A'_2 \left(\frac{y}{x+y} \right) + A_2 \left(\frac{y}{x+y} \right) \right] + \frac{xy}{(x+y)^2} A''_2 \left(\frac{y}{x+y} \right) \right\} \\ \cdot \exp \left\{ -(x+y) A_2 \left(\frac{y}{x+y} \right) \right\}, & \text{if } w_1 < \frac{y}{x+y} \leq w_2, \\ \vdots \\ \left\{ \left[\frac{y}{x+y} A'_p \left(\frac{y}{x+y} \right) - A_p \left(\frac{y}{x+y} \right) \right] \left[\frac{x}{x+y} A'_p \left(\frac{y}{x+y} \right) + A_p \left(\frac{y}{x+y} \right) \right] + \frac{xy}{(x+y)^2} A''_p \left(\frac{y}{x+y} \right) \right\} \\ \cdot \exp \left\{ -(x+y) A_p \left(\frac{y}{x+y} \right) \right\}, & \text{if } w_{p-1} \leq \frac{y}{x+y} \leq 1. \end{cases}$$

(4 marks)

UNSEEN

Solutions to Question B1

a) The cumulative distribution function of T conditional on $N = n$ is

$$\begin{aligned}\Pr(T \leq t \mid N = n) &= \Pr(\max(X_1, \dots, X_N) \leq t \mid N = n) \\ &= \Pr(\max(X_1, \dots, X_n) \leq t \mid N = n) \\ &= \Pr(X_1 \leq t, \dots, X_n \leq t) \\ &= \Pr(X_1 \leq t) \cdots \Pr(X_n \leq t) \\ &= [1 + \exp(-t)]^{-1} \cdots [1 + \exp(-t)]^{-1} \\ &= [1 + \exp(-t)]^{-n}.\end{aligned}$$

(4 marks)

UNSEEN

b) The unconditional cumulative distribution function of T is

$$\begin{aligned}\Pr(T \leq t) &= \sum_{n=1}^{\infty} \Pr(T \leq t \mid N = n) \theta (1 - \theta)^{n-1} \\ &= \sum_{n=1}^{\infty} [1 + \exp(-t)]^{-n} \theta (1 - \theta)^{n-1} \\ &= \theta [1 + \exp(-t)]^{-1} \sum_{n=1}^{\infty} [1 + \exp(-t)]^{1-n} (1 - \theta)^{n-1} \\ &= \theta [1 + \exp(-t)]^{-1} \{1 - [1 + \exp(-t)]^{-1} (1 - \theta)\}^{-1} \\ &= \frac{\theta}{\theta + \exp(-t)}.\end{aligned}$$

(4 marks)

UNSEEN

c) The unconditional probability density function of T is

$$f_T(t) = \frac{\theta \exp(-t)}{[\theta + \exp(-t)]^2}.$$

(1 marks)

UNSEEN

d) The moment generating function of T is

$$\begin{aligned}
M_T(s) &= \int_{-\infty}^{\infty} \frac{\theta \exp(st - t)}{[\theta + \exp(-t)]^2} dt \\
&= \theta \int_{-\infty}^{\infty} \frac{\exp(st - t)}{[\theta + \exp(-t)]^2} dt \\
&= \theta^{-s} \int_0^1 y^{1+s} (1-y)^{1-s} \left(\frac{1}{1-y} - \frac{1}{y} \right) dy \\
&= \theta^{-s} \left[\int_0^1 y^{1+s} (1-y)^{-s} dy - \int_0^1 y^s (1-y)^{1-s} dy \right] \\
&= \theta^{-s} [B(2+s, 1-s) - B(1+s, 2-s)],
\end{aligned}$$

where we have set $y = \frac{\theta}{\theta + \exp(-t)}$ and used the definition of the beta function.

(4 marks)

UNSEEN

e) Setting

$$\frac{\theta}{\theta + \exp(-t)} = p$$

and solving for t , we obtain value at risk of T as

$$\text{VaR}_p(T) = -\log \left[\theta \frac{1-p}{p} \right].$$

(3 marks)

UNSEEN

f) The expected shortfall T is

$$\begin{aligned}
\text{ES}_p(T) &= \frac{1}{p} \int_0^p [-\log \theta - \log(1-u) + \log u] du \\
&= -\log \theta - \frac{1}{p} \int_0^p \log(1-u) du + \frac{1}{p} \int_0^p \log u du \\
&= -\log \theta - \frac{1}{p} \left\{ [u \log(1-u)]_0^p + \int_0^p \frac{u}{1-u} du \right\} + \frac{1}{p} \left\{ [u \log u]_0^p - \int_0^p 1 du \right\} \\
&= -\log \theta - \frac{1}{p} \left\{ p \log(1-p) + \int_0^p \frac{u-1+1}{1-u} du \right\} + \frac{1}{p} \{p \log p - p\} \\
&= -\log \theta - \frac{1}{p} \{p \log(1-p) - p - \log(1-p)\} + \log p - 1 \\
&= -\log \theta + \log \frac{p}{1-p} + \frac{\log(1-p)}{p}.
\end{aligned}$$

(4 marks)

UNSEEN

Solutions to Question B2

If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cdf of a normalized version of M_n converges to G , i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n(a_n x + b_n) \rightarrow G(x) \quad (1)$$

as $n \rightarrow \infty$ then G must be of the same type as (cdf's G and G^* are of the same type if $G^*(x) = G(ax + b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$\begin{aligned} I & : \Lambda(x) = \exp\{-\exp(-x)\}, \quad x \in \mathfrak{R}; \\ II & : \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\{-x^{-\alpha}\} & \text{if } x \geq 0 \end{cases} \\ & \quad \text{for some } \alpha > 0; \\ III & : \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^\alpha\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \\ & \quad \text{for some } \alpha > 0. \end{aligned}$$

(4 marks)

SEEN

The necessary and sufficient conditions for the three extreme value distributions are:

$$\begin{aligned} I & : \exists \gamma(t) > 0 \text{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = \exp(-x), \quad x \in \mathfrak{R}, \\ II & : w(F) = \infty \text{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \quad x > 0, \\ III & : w(F) < \infty \text{ and } \lim_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} = x^\alpha, \quad x > 0. \end{aligned}$$

(4 marks)

SEEN

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function say $h(t)$ such that

$$\lim_{t \rightarrow w(G)} \frac{1 - G(t + xh(t))}{1 - G(t)} = e^{-x}$$

for every $x > 0$. But

$$\begin{aligned}
& \lim_{t \rightarrow w(F)} \frac{1 - F(t + xh(t))}{1 - F(t)} \\
= & \lim_{t \rightarrow w(F)} \frac{(1 + xh'(t)) f(t + xh(t))}{f(t)} \\
= & \lim_{t \rightarrow w(G)} \frac{(1 + xh'(t)) g(t + xh(t)) [1 - G(t + xh(t))]^{\lambda b - 1} \left\{ 1 - [1 - G(t + xh(t))]^\lambda \right\}^{a-1}}{g(t) [1 - G(t)]^{\lambda b - 1} \left\{ 1 - [1 - G(t)]^\lambda \right\}^{a-1}} \\
= & \lim_{t \rightarrow w(G)} \frac{(1 + xh'(t)) g(t + xh(t)) \left[\frac{1 - G(t + xh(t))}{1 - G(t)} \right]^{\lambda b - 1} \left\{ \frac{1 - [1 - G(t + xh(t))]^\lambda}{1 - [1 - G(t)]^\lambda} \right\}^{a-1}}{g(t)} \\
= & \lim_{t \rightarrow w(G)} \frac{1 - G(t + xh(t))}{1 - G(t)} \left[\frac{1 - G(t + xh(t))}{1 - G(t)} \right]^{\lambda b - 1} \left\{ \frac{1 - [1 - G(t + xh(t))]^\lambda}{1 - [1 - G(t)]^\lambda} \right\}^{a-1} \\
= & \lim_{t \rightarrow w(G)} \left[\frac{1 - G(t + xh(t))}{1 - G(t)} \right]^{\lambda b} \left\{ \frac{1 - [1 - G(t + xh(t))]^\lambda}{1 - [1 - G(t)]^\lambda} \right\}^{a-1} \\
= & \lim_{t \rightarrow w(G)} \left[\frac{1 - G(t + xh(t))}{1 - G(t)} \right]^{\lambda b} \left\{ \frac{1 - [1 - \lambda G(t + xh(t))]}{1 - [1 - \lambda G(t)]} \right\}^{a-1} \\
= & \lim_{t \rightarrow w(G)} \left[\frac{1 - G(t + xh(t))}{1 - G(t)} \right]^{\lambda b} \left\{ \frac{G(t + xh(t))}{G(t)} \right\}^{a-1} \\
= & \lim_{t \rightarrow w(G)} \left[\frac{1 - G(t + xh(t))}{1 - G(t)} \right]^{\lambda b} \\
= & \exp(-\lambda bx)
\end{aligned}$$

for every $x > 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n \rightarrow \infty} P \left(\frac{M_n - b_n}{a_n} \leq x \right) = \exp[-\exp(-\lambda bx)]$$

for some suitable norming constants $a_n > 0$ and b_n .

(4 marks)

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t \rightarrow \infty} \frac{1 - G(tx)}{1 - G(t)} = x^{-\beta}$$

for every $x > 0$. But

$$\begin{aligned}
& \lim_{t \rightarrow \infty} \frac{1 - F(tx)}{1 - F(t)} \\
= & \lim_{t \rightarrow \infty} \frac{x f(tx)}{f(t)} \\
= & \lim_{t \rightarrow \infty} \frac{xg(tx) [1 - G(tx)]^{\lambda b - 1} \left\{ 1 - [1 - G(tx)]^\lambda \right\}^{a-1}}{g(t) [1 - G(t)]^{\lambda b - 1} \left\{ 1 - [1 - G(t)]^\lambda \right\}^{a-1}} \\
= & \lim_{t \rightarrow \infty} \frac{xg(tx)}{g(t)} \left[\frac{1 - G(tx)}{1 - G(t)} \right]^{\lambda b - 1} \left\{ \frac{1 - [1 - G(tx)]^\lambda}{1 - [1 - G(t)]^\lambda} \right\}^{a-1} \\
= & \lim_{t \rightarrow \infty} \frac{1 - G(tx)}{1 - G(t)} \left[\frac{1 - G(tx)}{1 - G(t)} \right]^{\lambda b - 1} \left\{ \frac{1 - [1 - G(tx)]^\lambda}{1 - [1 - G(t)]^\lambda} \right\}^{a-1} \\
= & \lim_{t \rightarrow \infty} \left[\frac{1 - G(tx)}{1 - G(t)} \right]^{\lambda b} \left\{ \frac{1 - [1 - G(tx)]^\lambda}{1 - [1 - G(t)]^\lambda} \right\}^{a-1} \\
= & \lim_{t \rightarrow \infty} \left[\frac{1 - G(tx)}{1 - G(t)} \right]^{\lambda b} \left\{ \frac{1 - [1 - \lambda G(tx)]}{1 - [1 - \lambda G(t)]} \right\}^{a-1} \\
= & \lim_{t \rightarrow \infty} \left[\frac{1 - G(tx)}{1 - G(t)} \right]^{\lambda b} \left\{ \frac{G(tx)}{G(t)} \right\}^{a-1} \\
= & \lim_{t \rightarrow \infty} \left[\frac{1 - G(tx)}{1 - G(t)} \right]^{\lambda b} \\
= & x^{-\lambda b \beta}
\end{aligned}$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n \rightarrow \infty} P \left(\frac{M_n - b_n}{a_n} \leq x \right) = \exp(-x^{-\lambda b \beta})$$

for some suitable norming constants $a_n > 0$ and b_n .

(4 marks)

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t \rightarrow 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} = x^\beta$$

for every $x > 0$.

$$\begin{aligned}
& \lim_{t \rightarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} \\
&= \lim_{t \rightarrow 0} \frac{xf(w(F) - tx)}{f(w(F) - t)} \\
&= \lim_{t \rightarrow 0} \frac{xg(w(F) - tx) [1 - G(w(F) - tx)]^{\lambda b - 1} \left\{ 1 - [1 - G(w(F) - tx)]^\lambda \right\}^{a-1}}{g(w(F) - t) [1 - G(w(F) - t)]^{\lambda b - 1} \left\{ 1 - [1 - G(w(F) - t)]^\lambda \right\}^{a-1}} \\
&= \lim_{t \rightarrow 0} \frac{xg(w(F) - tx)}{g(w(F) - t)} \left[\frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \right]^{\lambda b - 1} \left\{ \frac{1 - [1 - G(w(F) - tx)]^\lambda}{1 - [1 - G(w(F) - t)]^\lambda} \right\}^{a-1} \\
&= \lim_{t \rightarrow 0} \frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \left[\frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \right]^{\lambda b - 1} \left\{ \frac{1 - [1 - G(w(F) - tx)]^\lambda}{1 - [1 - G(w(F) - t)]^\lambda} \right\}^{a-1} \\
&= \lim_{t \rightarrow 0} \left[\frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \right]^{\lambda b} \left\{ \frac{1 - [1 - G(w(F) - tx)]^\lambda}{1 - [1 - G(w(F) - t)]^\lambda} \right\}^{a-1} \\
&= \lim_{t \rightarrow 0} \left[\frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \right]^{\lambda b} \left\{ \frac{1 - [1 - \lambda G(w(F) - tx)]}{1 - [1 - \lambda G(w(F) - t)]} \right\}^{a-1} \\
&= \lim_{t \rightarrow 0} \left[\frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \right]^{\lambda b} \left\{ \frac{G(w(F) - tx)}{G(w(F) - t)} \right\}^{a-1} \\
&= \lim_{t \rightarrow 0} \left[\frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \right]^{\lambda b} \\
&= x^{\lambda b \beta}
\end{aligned}$$

for every $x < 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{n \rightarrow \infty} P \left(\frac{M_n - b_n}{a_n} \leq x \right) = \exp(-(-x)^{\lambda b \beta})$$

for some suitable norming constants $a_n > 0$ and b_n .

(4 marks)

UNSEEN

Solutions to Question B3

a) Note that $w(F) = n$ and

$$\begin{aligned}
 \frac{\Pr(X = w(F))}{1 - F(w(F) - 1)} &= \frac{\Pr(X = n)}{1 - F(n - 1)} \\
 &= \frac{\Pr(X = n)}{1 - \Pr(X = 1) - \Pr(X = 2) - \dots - \Pr(X = n - 1)} \\
 &= \frac{\Pr(X = n)}{\Pr(X = n)} \\
 &= 1.
 \end{aligned}$$

Hence, there can be no non-degenerate limit.

(4 marks)

UNSEEN

b) Note that $w(F) = \min(n, K)$ and

$$\begin{aligned}
 &\frac{\Pr(X = w(F))}{1 - F(w(F) - 1)} \\
 = &\frac{\Pr(X = \min(n, K))}{1 - F(\min(n, K) - 1)} \\
 = &\frac{\Pr(X = \min(n, K))}{1 - \Pr(X = \max(0, n + K - N)) - \Pr(X = \max(0, n + K - N) + 1) - \dots - \Pr(X = \min(n, K) - 1)} \\
 = &\frac{\Pr(X = \min(n, K))}{\Pr(X = \min(n, K))} \\
 = &1.
 \end{aligned}$$

Hence, there can be no non-degenerate limit.

(4 marks)

UNSEEN

c) Note that $w(F) = \infty$. Note that

$$\begin{aligned}
\lim_{t \downarrow \infty} \frac{1 - F(t + xh(t))}{1 - F(t)} &= \lim_{t \downarrow \infty} \frac{(1 + xh'(t)) f(t + xh(t))}{f(t)} \\
&= \lim_{t \downarrow \infty} \frac{(1 + xh'(t)) \exp[b(t + xh(t)) - \eta \exp(bt + bxh(t))]}{\exp[bt - \eta \exp(bt)]} \\
&= \lim_{t \downarrow \infty} (1 + xh'(t)) \exp\{bxh(t) + \eta \exp(bt) [1 - \exp(bxh(t))]\} \\
&= \lim_{t \downarrow \infty} (1 + xh'(t)) \exp\{bxh(t) - \eta \exp(bt)bxh(t)\} \\
&= \exp(-x)
\end{aligned}$$

if $h(t) = \frac{1}{\eta b \exp(bt)}$. So, $F(x)$ belongs to the Gumbel domain of attraction.

(4 marks)

UNSEEN

d) Note that $w(F) = \infty$. Then

$$\begin{aligned}
\lim_{t \rightarrow \infty} \frac{1 - F(tx)}{1 - F(t)} &= \lim_{t \rightarrow \infty} \frac{xf(tx)}{f(t)} \\
&= \lim_{t \rightarrow \infty} \frac{x(tx)^{-\alpha-1} \exp(-\frac{\beta}{tx})}{t^{-\alpha-1} \exp(-\frac{\beta}{t})} \\
&= \lim_{t \rightarrow \infty} \frac{x(tx)^{-\alpha-1}}{t^{-\alpha-1}} \\
&= x^{-\alpha}.
\end{aligned}$$

So, $F(x)$ belongs to the Fréchet domain of attraction.

(4 marks)

UNSEEN

e) Note that $w(F) = \infty$. Then

$$\begin{aligned}
\lim_{t \rightarrow \infty} \frac{1 - F(t + xh(t))}{1 - F(t)} &= \lim_{t \rightarrow \infty} \frac{1 - [1 + \exp(-at - axh(t))]^{-b}}{1 - [1 + \exp(-at)]^{-b}} \\
&= \lim_{t \rightarrow \infty} \frac{1 - [1 - b \exp(-at - axh(t))]}{1 - [1 - b \exp(-at)]} \\
&= \lim_{t \rightarrow \infty} \frac{\exp(-at - axh(t))}{\exp(-at)} \\
&= \lim_{t \rightarrow \infty} \exp(-axh(t)) \\
&= \exp(-x)
\end{aligned}$$

if $h(t) = \frac{1}{a}$. So, the cdf $F(x)$ belongs to the Gumbel domain of attraction.

(4 marks)

UNSEEN

Solutions to Question B4

(a) If X is an absolutely continuous random variable with cdf $F(\cdot)$ then

$$\text{VaR}_p(X) = F^{-1}(p)$$

and

$$\text{ES}_p(X) = \frac{1}{p} \int_0^p F^{-1}(v) dv.$$

(2 marks)

SEEN

(b) (i) T is a $N(\mu_1, \sigma_1^2) + \dots + N(\mu_k, \sigma_k^2) \equiv N(\mu_1 + \dots + \mu_k, \sigma_1^2 + \dots + \sigma_k^2)$ random variable;

(2 marks)

UNSEEN

(b) (ii) Inverting

$$\Phi\left(\frac{t - \mu_1 - \dots - \mu_k}{\sqrt{\sigma_1^2 + \dots + \sigma_k^2}}\right) = p,$$

we obtain

$$\text{VaR}_p(T) = \mu_1 + \dots + \mu_k + \sqrt{\sigma_1^2 + \dots + \sigma_k^2} \Phi^{-1}(p).$$

(2 marks)

UNSEEN

(b) (iii) The expected shortfall is

$$\begin{aligned} \text{ES}_p(T) &= \frac{1}{p} \int_0^p \left[\mu_1 + \dots + \mu_k + \sqrt{\sigma_1^2 + \dots + \sigma_k^2} \Phi^{-1}(v) \right] dv \\ &= \mu_1 + \dots + \mu_k + \sqrt{\sigma_1^2 + \dots + \sigma_k^2} \frac{1}{p} \int_0^p \Phi^{-1}(v) dv. \end{aligned}$$

(2 marks)

UNSEEN

c) (i) The joint likelihood function of $\mu_1, \mu_2, \dots, \mu_k$ and $\sigma_1^2, \sigma_2^2, \dots, \sigma_k^2$ is

$$\begin{aligned} & L(\mu_1, \mu_2, \dots, \mu_k, \sigma_1^2, \sigma_2^2, \dots, \sigma_k^2) \\ &= \prod_{i=1}^k \prod_{j=1}^n \left\{ \frac{1}{\sqrt{2\pi}\sigma_i} \exp \left[-\frac{(X_{i,j} - \mu_i)^2}{2\sigma_i^2} \right] \right\} \\ &= \prod_{i=1}^k \left\{ \frac{1}{(2\pi)^n \sigma_i^n} \exp \left[-\frac{1}{2\sigma_i^2} \sum_{j=1}^n (X_{i,j} - \mu_i)^2 \right] \right\} \\ &= \frac{1}{(2\pi)^{nk} \sigma_1^n \dots \sigma_k^n} \exp \left[-\frac{1}{2} \sum_{i=1}^k \frac{1}{\sigma_i^2} \sum_{j=1}^n (X_{i,j} - \mu_i)^2 \right]. \end{aligned}$$

(2 marks)

UNSEEN

c) (ii) The log likelihood function is

$$\log L = -nk \log(2\pi) - n \sum_{i=1}^k \log \sigma_i - \frac{1}{2} \sum_{i=1}^k \frac{1}{\sigma_i^2} \sum_{j=1}^n (X_{i,j} - \mu_i)^2.$$

The partial derivatives are

$$\frac{\partial \log L}{\partial \mu_i} = \frac{1}{\sigma_i^2} \sum_{j=1}^n (X_{i,j} - \mu_i) = \frac{1}{\sigma_i^2} \left[\left(\sum_{j=1}^n X_{i,j} \right) - n\mu_i \right]$$

and

$$\frac{\partial \log L}{\partial \sigma_i} = -\frac{n}{\sigma_i} + \frac{1}{\sigma_i^3} \sum_{j=1}^n (X_{i,j} - \mu_i)^2.$$

Setting these to zero and solving, we obtain

$$\hat{\mu}_i = \frac{1}{n} \sum_{j=1}^n X_{i,j}$$

and

$$\hat{\sigma}_i^2 = \frac{1}{n} \sum_{j=1}^n (X_{i,j} - \hat{\mu}_i)^2.$$

(4 marks)

UNSEEN

c) (iii) $\widehat{\mu}_i$ is unbiased and consistent since

$$E(\widehat{\mu}_i) = \frac{1}{n} \sum_{j=1}^n E(X_{i,j}) = \frac{1}{n} \sum_{j=1}^n \mu_i = \mu_i$$

and

$$Var(\widehat{\mu}_i) = \frac{1}{n^2} \sum_{j=1}^n Var(X_{i,j}) = \frac{1}{n^2} \sum_{j=1}^n \sigma_i^2 = \frac{\sigma_i^2}{n}.$$

(2 marks)

UNSEEN

c) (iv) $\widehat{\sigma}_i^2$ is unbiased and consistent since

$$E(\widehat{\sigma}_i^2) = \frac{\sigma_i^2}{n} E\left[\frac{1}{\sigma_i^2} \sum_{j=1}^n (X_{i,j} - \widehat{\mu}_i)^2\right] = \frac{\sigma_i^2}{n} E[\chi_{n-1}^2] = \frac{(n-1)\sigma_i^2}{n}$$

and

$$Var(\widehat{\sigma}_i^2) = Var\left[\frac{\sigma_i^2}{n} \frac{1}{\sigma_i^2} \sum_{j=1}^n (X_{i,j} - \widehat{\mu}_i)^2\right] = \frac{\sigma_i^4}{n^2} Var\left[\frac{1}{\sigma_i^2} \sum_{j=1}^n (X_{i,j} - \widehat{\mu}_i)^2\right] = \frac{\sigma_i^4}{n^2} Var[\chi_{n-1}^2] = \frac{2\sigma_i^4(n-1)}{n^2}.$$

(2 marks)

UNSEEN

c) (v) The maximum likelihood estimators of $VaR_p(T)$ and $ES_p(T)$ are

$$\widehat{VaR}_p(T) = \frac{1}{n} \sum_{i=1}^k \sum_{j=1}^n X_{i,j} + \sqrt{\frac{1}{n} \sum_{i=1}^k \sum_{j=1}^n (X_{i,j} - \widehat{\mu}_i)^2 \Phi^{-1}(p)}$$

and

$$\widehat{ES}_p(T) = \frac{1}{n} \sum_{i=1}^k \sum_{j=1}^n X_{i,j} + \sqrt{\frac{1}{n} \sum_{i=1}^k \sum_{j=1}^n (X_{i,j} - \widehat{\mu}_i)^2 \frac{1}{p} \int_0^p \Phi^{-1}(v) dv}.$$

(2 marks)

UNSEEN

Solutions to Question B5

a) Note that

$$\begin{aligned}F_T(t) &= \Pr(T \leq t) \\&= 1 - \Pr(T > t) \\&= 1 - \Pr(\min(X_1, X_2, \dots, X_k) > t) \\&= 1 - \Pr(X_1 > t, X_2 > t, \dots, X_k > t) \\&= 1 - \bar{F}(t, t, \dots, t) \\&= 1 - \left[\max\left(\frac{t}{a_1}, \frac{t}{a_2}, \dots, \frac{t}{a_k}\right) \right]^{-a} \\&= 1 - \left[\max\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_k}\right) \right]^{-a} t^{-a} \\&= 1 - [\min(a_1, a_2, \dots, a_k)]^a t^{-a}\end{aligned}$$

for $t > \min(a_1, a_2, \dots, a_k)$.

(6 marks)

UNSEEN

b) The corresponding pdf is

$$f_T(t) = a [\min(a_1, a_2, \dots, a_k)]^a t^{-a-1}$$

for $t > \min(a_1, a_2, \dots, a_k)$.

(2 marks)

UNSEEN

c) Inverting

$$1 - [\min(a_1, a_2, \dots, a_k)]^a t^{-a} = p$$

gives

$$\text{VaR}_p(T) = \min(a_1, a_2, \dots, a_k) (1 - p)^{-1/a}.$$

(2 marks)

UNSEEN

d) The expected shortfall is

$$\begin{aligned} \text{ES}_p(T) &= \frac{\min(a_1, a_2, \dots, a_k)}{p} \int_0^p (1-u)^{-\frac{1}{a}} du \\ &= \frac{\min(a_1, a_2, \dots, a_k)}{p \left(\frac{1}{a} - 1\right)} \left[(1-u)^{1-\frac{1}{a}} \right]_0^p \\ &= \frac{\min(a_1, a_2, \dots, a_k)}{p \left(\frac{1}{a} - 1\right)} \left[(1-p)^{1-\frac{1}{a}} - 1 \right]. \end{aligned}$$

(2 marks)

UNSEEN

e) The likelihood function is

$$\begin{aligned} L(a, a_1, \dots, a_k) &= a^n [\min(a_1, a_2, \dots, a_k)]^{na} \left(\prod_{i=1}^n t_i \right)^{-a-1} \left\{ \prod_{i=1}^n I[t_i > \min(a_1, \dots, a_k)] \right\} \\ &= a^n [\min(a_1, a_2, \dots, a_k)]^{na} \left(\prod_{i=1}^n t_i \right)^{-a-1} \{ I[\min(t_1, \dots, t_n) > \min(a_1, \dots, a_k)] \}. \end{aligned}$$

As a function of $\min(a_1, \dots, a_k)$, it is increasing over $(-\infty, \min(t_1, \dots, t_n))$. Hence, the maximum likelihood estimator of a_1, a_2, \dots, a_k are those values satisfying $\min(a_1, \dots, a_k) = \min(t_1, \dots, t_n)$.

To find the maximum likelihood estimator of a , take the log of the likelihood

$$\log L(a, a_1, \dots, a_k) = n \log a + na \log [\min(a_1, a_2, \dots, a_k)] - (a+1) \sum_{i=1}^n \log t_i.$$

The partial derivative with respect to a is

$$\frac{\partial \log L}{\partial a} = \frac{n}{a} + n \log [\min(a_1, a_2, \dots, a_k)] - \sum_{i=1}^n \log t_i.$$

Setting this to zero gives

$$\hat{a} = n \left\{ -n \log [\min(a_1, a_2, \dots, a_k)] + \sum_{i=1}^n \log t_i \right\}^{-1}.$$

This is a maximum likelihood estimator since $\frac{\partial^2 \log L}{\partial a^2} = -\frac{n}{a^2} < 0$.

(8 marks)

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