SOLUTIONS TO MATH38181 EXTREME VALUES AND FINANCIAL RISK EXAM

Solutions to Question 1

a) The cumulative distribution function of T conditional on $N = n$ is

$$\begin{aligned}
\Pr(T \le t \mid N = n) &= \Pr(\max(X_1, \ldots, X_N) \le t \mid N = n) \\
&= \Pr(\max(X_1, \ldots, X_n) \le t \mid N = n) \\
&= \Pr(X_1 \le t, \ldots, X_n \le t) \\
&= \Pr(X_1 \le t) \cdots \Pr(X_n \le t) \\
&= [1 + \exp(-t)]^{-1} \cdots [1 + \exp(-t)]^{-1} \\
&= [1 + \exp(-t)]^{-n}.
\end{aligned}$$

(4 marks)

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b) The unconditional cumulative distribution function of T is

$$\begin{aligned}
\Pr(T \le t) &= \sum_{n=1}^{\infty} \Pr(T \le t \mid N = n)\, \theta (1-\theta)^{n-1} \\
&= \sum_{n=1}^{\infty} [1 + \exp(-t)]^{-n}\, \theta (1-\theta)^{n-1} \\
&= \theta\, [1 + \exp(-t)]^{-1} \sum_{n=1}^{\infty} [1 + \exp(-t)]^{1-n} (1-\theta)^{n-1} \\
&= \theta\, [1 + \exp(-t)]^{-1} \left\{ 1 - [1 + \exp(-t)]^{-1} (1-\theta) \right\}^{-1} \\
&= \frac{\theta}{\theta + \exp(-t)}.
\end{aligned}$$

(4 marks)

UNSEEN

c) The unconditional probability density function of T is

$$f_T(t) = \frac{\theta \exp(-t)}{[\theta + \exp(-t)]^2}.$$

(1 marks)

UNSEEN

d) The moment generating function of T is

$$\begin{aligned}
M_T(s) &= \int_{-\infty}^{\infty} \frac{\theta \exp(st-t)}{[\theta+\exp(-t)]^2} dt \\
&= \theta \int_{-\infty}^{\infty} \frac{\exp(st-t)}{[\theta+\exp(-t)]^2} dt \\
&= \theta^{-s} \int_0^1 y^{1+s}(1-y)^{1-s} \left(\frac{1}{1-y} - \frac{1}{y}\right) dy \\
&= \theta^{-s} \left[\int_0^1 y^{1+s}(1-y)^{-s} dy - \int_0^1 y^{s}(1-y)^{1-s} dy\right] \\
&= \theta^{-s} \left[B(2+s, 1-s) - B(1+s, 2-s)\right],
\end{aligned}$$

where we have set $y = \frac{\theta}{\theta+\exp(-t)}$ and used the definition of the beta function.

(4 marks)

UNSEEN

e) Setting

$$\frac{\theta}{\theta+\exp(-t)} = p$$

and solving for t, we obtain value at risk of T as

$$\mathrm{VaR}_p(T) = -\log\left[\theta \frac{1-p}{p}\right].$$

(3 marks)

UNSEEN

f) The expected shortfall T is

$$\begin{aligned}
\mathrm{ES}_p(T) &= \frac{1}{p}\int_0^p \left[-\log\theta - \log(1-u) + \log u\right] du \\
&= -\log\theta - \frac{1}{p}\int_0^p \log(1-u) du + \frac{1}{p}\int_0^p \log u du \\
&= -\log\theta - \frac{1}{p}\left\{[u\log(1-u)]_0^p + \int_0^p \frac{u}{1-u} du\right\} + \frac{1}{p}\left\{[u\log u]_0^p - \int_0^p 1 du\right\} \\
&= -\log\theta - \frac{1}{p}\left\{p\log(1-p) + \int_0^p \frac{u-1+1}{1-u} du\right\} + \frac{1}{p}\left\{p\log p - p\right\} \\
&= -\log\theta - \frac{1}{p}\left\{p\log(1-p) - p - \log(1-p)\right\} + \log p - 1 \\
&= -\log\theta + \log\frac{p}{1-p} + \frac{\log(1-p)}{p}.
\end{aligned}$$

(4 marks)

UNSEEN

Solutions to Question 2

If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cdf of a normalized version of M_n converges to G, i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n\left(a_n x + b_n\right) \to G(x) \tag{1}$$

as $n \to \infty$ then G must be of the same type as (cdfs G and G^* are of the same type if $G^*(x) = G(ax + b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$\begin{aligned}
I &: \Lambda(x) = \exp\left\{-\exp(-x)\right\}, \qquad x \in \Re; \\
II &: \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\left\{-x^{-\alpha}\right\} & \text{if } x \geq 0 \end{cases} \\
&\quad \text{for some } \alpha > 0; \\
III &: \Psi_\alpha(x) = \begin{cases} \exp\left\{-(-x)^{\alpha}\right\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \\
&\quad \text{for some } \alpha > 0.
\end{aligned}$$

(4 marks)

SEEN

The necessary and sufficient conditions for the three extreme value distributions are:

$$\begin{aligned}
I &: \exists \gamma(t) > 0 \text{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F\left(t + x\gamma(t)\right)}{1 - F(t)} = \exp(-x), \qquad x \in \Re, \\
II &: w(F) = \infty \text{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \qquad x > 0, \\
III &: w(F) < \infty \text{ and } \lim_{t \downarrow 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} = x^{\alpha}, \qquad x > 0.
\end{aligned}$$

(4 marks)

SEEN

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function say $h(t)$ such that

$$\lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} = e^{-x}$$

for every $x > 0$. But

$$
\begin{aligned}
& \lim_{t\to w(F)} \frac{1-F\left(t+xh(t)\right)}{1-F(t)} \\
= & \lim_{t\to w(F)} \frac{\left(1+xh^{'}(t)\right) f\left(t+xh(t)\right)}{f(t)} \\
= & \lim_{t\to w(G)} \frac{\left(1+xh^{'}(t)\right) g\left(t+xh(t)\right)\left[1-G\left(t+xh(t)\right)\right]^{\lambda b-1}\left\{1-\left[1-G\left(t+xh(t)\right)\right]^{\lambda}\right\}^{a-1}}{g\left(t\right)\left[1-G\left(t\right)\right]^{\lambda b-1}\left\{1-\left[1-G\left(t\right)\right]^{\lambda}\right\}^{a-1}} \\
= & \lim_{t\to w(G)} \frac{\left(1+xh^{'}(t)\right) g\left(t+xh(t)\right)}{g(t)} \left[\frac{1-G\left(t+xh(t)\right)}{1-G\left(t\right)}\right]^{\lambda b-1} \left\{\frac{1-\left[1-G\left(t+xh(t)\right)\right]^{\lambda}}{1-\left[1-G\left(t\right)\right]^{\lambda}}\right\}^{a-1} \\
= & \lim_{t\to w(G)} \frac{1-G\left(t+xh(t)\right)}{1-G(t)} \left[\frac{1-G\left(t+xh(t)\right)}{1-G\left(t\right)}\right]^{\lambda b-1} \left\{\frac{1-\left[1-G\left(t+xh(t)\right)\right]^{\lambda}}{1-\left[1-G\left(t\right)\right]^{\lambda}}\right\}^{a-1} \\
= & \lim_{t\to w(G)} \left[\frac{1-G\left(t+xh(t)\right)}{1-G\left(t\right)}\right]^{\lambda b} \left\{\frac{1-\left[1-G\left(t+xh(t)\right)\right]^{\lambda}}{1-\left[1-G\left(t\right)\right]^{\lambda}}\right\}^{a-1} \\
= & \lim_{t\to w(G)} \left[\frac{1-G\left(t+xh(t)\right)}{1-G\left(t\right)}\right]^{\lambda b} \left\{\frac{1-\left[1-\lambda G\left(t+xh(t)\right)\right]}{1-\left[1-\lambda G\left(t\right)\right]}\right\}^{a-1} \\
= & \lim_{t\to w(G)} \left[\frac{1-G\left(t+xh(t)\right)}{1-G\left(t\right)}\right]^{\lambda b} \left\{\frac{G\left(t+xh(t)\right)}{G\left(t\right)}\right\}^{a-1} \\
= & \lim_{t\to w(G)} \left[\frac{1-G\left(t+xh(t)\right)}{1-G\left(t\right)}\right]^{\lambda b} \\
= & \exp(-\lambda b x)
\end{aligned}
$$

for every $x > 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left[-\exp(-\lambda b x)\right]$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t\to\infty} \frac{1-G\left(tx\right)}{1-G(t)} = x^{-\beta}$$

for every $x > 0$. But

$$\begin{aligned}
& \lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} \\
= & \lim_{t \to \infty} \frac{x f(tx)}{f(t)} \\
= & \lim_{t \to \infty} \frac{x g(tx) [1 - G(tx)]^{\lambda b - 1} \left\{ 1 - [1 - G(tx)]^{\lambda} \right\}^{a-1}}{g(t) [1 - G(t)]^{\lambda b - 1} \left\{ 1 - [1 - G(t)]^{\lambda} \right\}^{a-1}} \\
= & \lim_{t \to \infty} \frac{x g(tx)}{g(t)} \left[\frac{1 - G(tx)}{1 - G(t)} \right]^{\lambda b - 1} \left\{ \frac{1 - [1 - G(tx)]^{\lambda}}{1 - [1 - G(t)]^{\lambda}} \right\}^{a-1} \\
= & \lim_{t \to \infty} \frac{1 - G(tx)}{1 - G(t)} \left[\frac{1 - G(tx)}{1 - G(t)} \right]^{\lambda b - 1} \left\{ \frac{1 - [1 - G(tx)]^{\lambda}}{1 - [1 - G(t)]^{\lambda}} \right\}^{a-1} \\
= & \lim_{t \to \infty} \left[\frac{1 - G(tx)}{1 - G(t)} \right]^{\lambda b} \left\{ \frac{1 - [1 - G(tx)]^{\lambda}}{1 - [1 - G(t)]^{\lambda}} \right\}^{a-1} \\
= & \lim_{t \to \infty} \left[\frac{1 - G(tx)}{1 - G(t)} \right]^{\lambda b} \left\{ \frac{1 - [1 - \lambda G(tx)]}{1 - [1 - \lambda G(t)]} \right\}^{a-1} \\
= & \lim_{t \to \infty} \left[\frac{1 - G(tx)}{1 - G(t)} \right]^{\lambda b} \left\{ \frac{G(tx)}{G(t)} \right\}^{a-1} \\
= & \lim_{t \to \infty} \left[\frac{1 - G(tx)}{1 - G(t)} \right]^{\lambda b} \\
= & x^{-\lambda b \beta}
\end{aligned}$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n \to \infty} P \left(\frac{M_n - b_n}{a_n} \leq x \right) = \exp \left(-x^{-\lambda b \beta} \right)$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t \to 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} = x^{\beta}$$

for every $x > 0$.

$$
\begin{aligned}
& \lim_{t \to 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} \\
= \ & \lim_{t \to 0} \frac{xf\left(w(F) - tx\right)}{f\left(w(F) - t\right)} \\
= \ & \lim_{t \to 0} \frac{xg\left(w(F) - tx\right)\left[1 - G\left(w(F) - tx\right)\right]^{\lambda b - 1}\left\{1 - \left[1 - G\left(w(F) - tx\right)\right]^{\lambda}\right\}^{a-1}}{g\left(w(F) - t\right)\left[1 - G\left(w(F) - t\right)\right]^{\lambda b - 1}\left\{1 - \left[1 - G\left(w(F) - t\right)\right]^{\lambda}\right\}^{a-1}} \\
= \ & \lim_{t \to 0} \frac{xg\left(w(F) - tx\right)}{g\left(w(F) - t\right)}\left[\frac{1 - G\left(w(F) - tx\right)}{1 - G\left(w(F) - t\right)}\right]^{\lambda b - 1}\left\{\frac{1 - \left[1 - G\left(w(F) - tx\right)\right]^{\lambda}}{1 - \left[1 - G\left(w(F) - t\right)\right]^{\lambda}}\right\}^{a-1} \\
= \ & \lim_{t \to 0} \frac{1 - G\left(w(F) - tx\right)}{1 - G\left(w(F) - t\right)}\left[\frac{1 - G\left(w(F) - tx\right)}{1 - G\left(w(F) - t\right)}\right]^{\lambda b - 1}\left\{\frac{1 - \left[1 - G\left(w(F) - tx\right)\right]^{\lambda}}{1 - \left[1 - G\left(w(F) - t\right)\right]^{\lambda}}\right\}^{a-1} \\
= \ & \lim_{t \to 0} \left[\frac{1 - G\left(w(F) - tx\right)}{1 - G\left(w(F) - t\right)}\right]^{\lambda b}\left\{\frac{1 - \left[1 - G\left(w(F) - tx\right)\right]^{\lambda}}{1 - \left[1 - G\left(w(F) - t\right)\right]^{\lambda}}\right\}^{a-1} \\
= \ & \lim_{t \to 0} \left[\frac{1 - G\left(w(F) - tx\right)}{1 - G\left(w(F) - t\right)}\right]^{\lambda b}\left\{\frac{1 - \left[1 - \lambda G\left(w(F) - tx\right)\right]}{1 - \left[1 - \lambda G\left(w(F) - t\right)\right]}\right\}^{a-1} \\
= \ & \lim_{t \to 0} \left[\frac{1 - G\left(w(F) - tx\right)}{1 - G\left(w(F) - t\right)}\right]^{\lambda b}\left\{\frac{G\left(w(F) - tx\right)}{G\left(w(F) - t\right)}\right\}^{a-1} \\
= \ & \lim_{t \to 0} \left[\frac{1 - G\left(w(F) - tx\right)}{1 - G\left(w(F) - t\right)}\right]^{\lambda b} \\
= \ & x^{\lambda b \beta}
\end{aligned}
$$

for every $x < 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$
\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left(-(-x)^{\lambda b \beta}\right)
$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

UNSEEN

Solutions to Question 3

a) Note that $w(F) = n$ and

$$
\begin{aligned}
\frac{\Pr(X = w(F))}{1 - F(w(F) - 1)} &= \frac{\Pr(X = n)}{1 - F(n-1)} \\
&= \frac{\Pr(X = n)}{1 - \Pr(X = 1) - \Pr(X = 2) - \cdots - \Pr(X = n-1)} \\
&= \frac{\Pr(X = n)}{\Pr(X = n)} \\
&= 1.
\end{aligned}
$$

Hence, there can be no non-degenerate limit.

(4 marks)

UNSEEN

b) Note that $w(F) = \min(n, K)$ and

$$
\begin{aligned}
& \frac{\Pr(X = w(F))}{1 - F(w(F) - 1)} \\
= & \frac{\Pr(X = \min(n, K))}{1 - F(\min(n, K) - 1)} \\
= & \frac{\Pr(X = \min(n, K))}{1 - \Pr(X = \max(0, n + K - N)) - \Pr(X = \max(0, n + K - N) + 1) - \cdots - \Pr(X = \min(n, K) -} \\
= & \frac{\Pr(X = \min(n, K))}{\Pr(X = \min(n, K))} \\
= & 1.
\end{aligned}
$$

Hence, there can be no non-degenerate limit.

(4 marks)

UNSEEN

c) Note that $w(F) = \infty$. Note that

$$\begin{aligned}
\lim_{t \downarrow \infty} \frac{1 - F\left(t + xh(t)\right)}{1 - F\left(t\right)} &= \lim_{t \downarrow \infty} \frac{\left(1 + xh^{'}(t)\right) f\left(t + xh(t)\right)}{f\left(t\right)} \\
&= \lim_{t \downarrow \infty} \frac{\left(1 + xh^{'}(t)\right) \exp\left[b\left(t + xh(t)\right) - \eta \exp\left(bt + bxh(t)\right)\right]}{\exp\left[bt - \eta \exp(bt)\right]} \\
&= \lim_{t \downarrow \infty} \left(1 + xh^{'}(t)\right) \exp\left\{bxh(t) + \eta \exp(bt)\left[1 - \exp\left(bxh(t)\right)\right]\right\} \\
&= \lim_{t \downarrow \infty} \left(1 + xh^{'}(t)\right) \exp\left\{bxh(t) - \eta \exp(bt) bxh(t)\right\} \\
&= \exp(-x)
\end{aligned}$$

if $h(t) = \frac{1}{\eta b \exp(bt)}$. So, $F(x)$ belongs to the Gumbel domain of attraction.

(4 marks)

UNSEEN

d) Note that $w(F) = \infty$. Then

$$\begin{aligned}
\lim_{t \to \infty} \frac{1 - F\left(tx\right)}{1 - F(t)} &= \lim_{t \to \infty} \frac{xf\left(tx\right)}{f(t)} \\
&= \lim_{t \to \infty} \frac{x(tx)^{-\alpha-1} \exp\left(-\frac{\beta}{tx}\right)}{t^{-\alpha-1} \exp\left(-\frac{\beta}{t}\right)} \\
&= \lim_{t \to \infty} \frac{x(tx)^{-\alpha-1}}{t^{-\alpha-1}} \\
&= x^{-\alpha}.
\end{aligned}$$

So, $F(x)$ belongs to the Fréchet domain of attraction.

(4 marks)

UNSEEN

e) Note that $w(F) = \infty$. Then

$$\begin{aligned}
\lim_{t \to \infty} \frac{1 - F\left(t + xh(t)\right)}{1 - F(t)} &= \lim_{t \to \infty} \frac{1 - \left[1 + \exp\left(-at - axh(t)\right)\right]^{-b}}{1 - \left[1 + \exp\left(-at\right)\right]^{-b}} \\
&= \lim_{t \to \infty} \frac{1 - \left[1 - b \exp\left(-at - axh(t)\right)\right]}{1 - \left[1 - b \exp\left(-at\right)\right]} \\
&= \lim_{t \to \infty} \frac{\exp\left(-at - axh(t)\right)}{\exp\left(-at\right)} \\
&= \lim_{t \to \infty} \exp\left(-axh(t)\right) \\
&= \exp(-x)
\end{aligned}$$

if $h(t) = \frac{1}{a}$. So, the cdf $F(x)$ belongs to the Gumbel domain of attraction.

(4 marks)

UNSEEN

Solutions to Question 4

(a) If X is an absolutely continuous random variable with cdf $F(\cdot)$ then

$$\mathrm{VaR}_p(X) = F^{-1}(p)$$

and

$$\mathrm{ES}_p(X) = \frac{1}{p} \int_0^p F^{-1}(v) dv.$$

(2 marks)

SEEN

(b) (i) T is a $N\left(\mu_1, \sigma_1^2\right) + \cdots + N\left(\mu_k, \sigma_k^2\right) \equiv N\left(\mu_1 + \cdots + \mu_k, \sigma_1^2 + \cdots + \sigma_k^2\right)$ random variable;

(2 marks)

UNSEEN

(b) (ii) Inverting

$$\Phi\left(\frac{t - \mu_1 - \cdots - \mu_k}{\sqrt{\sigma_1^2 + \cdots + \sigma_k^2}}\right) = p,$$

we obtain

$$\mathrm{VaR}_p(T) = \mu_1 + \cdots + \mu_k + \sqrt{\sigma_1^2 + \cdots + \sigma_k^2}\,\Phi^{-1}(p).$$

(2 marks)

UNSEEN

(b) (iii) The expected shortfall is

$$\begin{aligned} \mathrm{ES}_p(T) &= \frac{1}{p} \int_0^p \left[\mu_1 + \cdots + \mu_k + \sqrt{\sigma_1^2 + \cdots + \sigma_k^2}\,\Phi^{-1}(v)\right] dv \\ &= \mu_1 + \cdots + \mu_k + \sqrt{\sigma_1^2 + \cdots + \sigma_k^2}\,\frac{1}{p} \int_0^p \Phi^{-1}(v) dv. \end{aligned}$$

(2 marks)

UNSEEN

c) (i) The joint likelihood function of $\mu_1, \mu_2, \ldots, \mu_k$ and $\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2$ is

$$
\begin{aligned}
& L\left(\mu_1, \mu_2, \ldots, \mu_k, \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2\right) \\
= \ & \prod_{i=1}^{k} \prod_{j=1}^{n} \left\{ \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left[-\frac{\left(X_{i,j} - \mu_i\right)^2}{2\sigma_i^2} \right] \right\} \\
= \ & \prod_{i=1}^{k} \left\{ \frac{1}{(2\pi)^n \sigma_i^n} \exp\left[-\frac{1}{2\sigma_i^2} \sum_{j=1}^{n} \left(X_{i,j} - \mu_i\right)^2 \right] \right\} \\
= \ & \frac{1}{(2\pi)^{nk} \sigma_1^n \cdots \sigma_k^n} \exp\left[-\frac{1}{2} \sum_{i=1}^{k} \frac{1}{\sigma_i^2} \sum_{j=1}^{n} \left(X_{i,j} - \mu_i\right)^2 \right].
\end{aligned}
$$

(2 marks)

UNSEEN

c) (ii) The log likelihood function is

$$
\log L = -nk \log(2\pi) - n \sum_{i=1}^{k} \log \sigma_i - \frac{1}{2} \sum_{i=1}^{k} \frac{1}{\sigma_i^2} \sum_{j=1}^{n} \left(X_{i,j} - \mu_i\right)^2 .
$$

The partial derivatives are

$$
\frac{\partial \log L}{\partial \mu_i} = \frac{1}{\sigma_i^2} \sum_{j=1}^{n} \left(X_{i,j} - \mu_i\right) = \frac{1}{\sigma_i^2} \left[\left(\sum_{j=1}^{n} X_{i,j} \right) - n\mu_i \right]
$$

and

$$
\frac{\partial \log L}{\partial \sigma_i} = -\frac{n}{\sigma_i} + \frac{1}{\sigma_i^3} \sum_{j=1}^{n} \left(X_{i,j} - \mu_i\right)^2 .
$$

Setting these to zero and solving, we obtain

$$
\widehat{\mu_i} = \frac{1}{n} \sum_{j=1}^{n} X_{i,j}
$$

and

$$
\widehat{\sigma_i^2} = \frac{1}{n} \sum_{j=1}^{n} \left(X_{i,j} - \widehat{\mu_i}\right)^2 .
$$

(4 marks)

UNSEEN

c) (iii) $\widehat{\mu_i}$ is unbiased and consistent since

$$E\left(\widehat{\mu_i}\right)=\frac{1}{n}\sum_{j=1}^{n}E\left(X_{i,j}\right)=\frac{1}{n}\sum_{j=1}^{n}\mu_i=\mu_i$$

and

$$Var\left(\widehat{\mu_i}\right)=\frac{1}{n^2}\sum_{j=1}^{n}Var\left(X_{i,j}\right)=\frac{1}{n^2}\sum_{j=1}^{n}\sigma_i^2=\frac{\sigma_i^2}{n}.$$

(2 marks)

UNSEEN

c) (iv) $\widehat{\sigma_i}^2$ is unbiased and consistent since

$$E\left(\widehat{\sigma_i}^2\right)=\frac{\sigma_i^2}{n}E\left[\frac{1}{\sigma_i^2}\sum_{j=1}^{n}\left(X_{i,j}-\widehat{\mu_i}\right)^2\right]=\frac{\sigma_i^2}{n}E\left[\chi_{n-1}^2\right]=\frac{(n-1)\sigma_i^2}{n}$$

and

$$Var\left(\widehat{\sigma_i}^2\right)=Var\left[\frac{\sigma_i^2}{n}\frac{1}{\sigma_i^2}\sum_{j=1}^{n}\left(X_{i,j}-\widehat{\mu_i}\right)^2\right]=\frac{\sigma_i^4}{n^2}Var\left[\frac{1}{\sigma_i^2}\sum_{j=1}^{n}\left(X_{i,j}-\widehat{\mu_i}\right)^2\right]=\frac{\sigma_i^4}{n^2}Var\left[\chi_{n-1}^2\right]=\frac{2\sigma_i^4(n-1)}{n^2}.$$

(2 marks)

UNSEEN

c (v) The maximum likelihood estimators of $\mathrm{VaR}_p(T)$ and $\mathrm{ES}_p(T)$ are

$$\widehat{\mathrm{VaR}}_p(T)=\frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n}X_{i,j}+\sqrt{\frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n}\left(X_{i,j}-\widehat{\mu_i}\right)^2}\Phi^{-1}(p)$$

and

$$\widehat{\mathrm{ES}}_p(T)=\frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n}X_{i,j}+\sqrt{\frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n}\left(X_{i,j}-\widehat{\mu_i}\right)^2}\frac{1}{p}\int_0^p\Phi^{-1}(v)dv.$$

(2 marks)

UNSEEN

Solutions to Question 5

a) Note that

$$\begin{aligned}
F_T(t) &= \Pr(T \leq t) \\
&= 1 - \Pr(T > t) \\
&= 1 - \Pr\left(\min\left(X_1, X_2, \ldots, X_k\right) > t\right) \\
&= 1 - \Pr\left(X_1 > t, X_2 > t, \ldots, X_k > t\right) \\
&= 1 - \overline{F}\left(t, t, \ldots, t\right) \\
&= 1 - \left[\max\left(\frac{t}{a_1}, \frac{t}{a_2}, \ldots, \frac{t}{a_k}\right)\right]^{-a} \\
&= 1 - \left[\max\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_k}\right)\right]^{-a} t^{-a} \\
&= 1 - \left[\min\left(a_1, a_2, \ldots, a_k\right)\right]^{a} t^{-a}
\end{aligned}$$

for $t > \min\left(a_1, a_2, \ldots, a_k\right)$.

(6 marks)

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b) The corresponding pdf is

$$f_T(t) = a \left[\min\left(a_1, a_2, \ldots, a_k\right)\right]^{a} t^{-a-1}$$

for $t > \min\left(a_1, a_2, \ldots, a_k\right)$.

(2 marks)

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c) Inverting

$$1 - \left[\min\left(a_1, a_2, \ldots, a_k\right)\right]^{a} t^{-a} = p$$

gives

$$\mathrm{VaR}_p(T) = \min\left(a_1, a_2, \ldots, a_k\right)(1-p)^{-1/a}.$$

(2 marks)

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d) The expected shortfall is

$$\begin{aligned}
\mathrm{ES}_p(T) &= \frac{\min\left(a_1, a_2, \ldots, a_k\right)}{p} \int_0^p (1-u)^{-\frac{1}{a}} du \\
&= \frac{\min\left(a_1, a_2, \ldots, a_k\right)}{p\left(\frac{1}{a}-1\right)} \left[(1-u)^{1-\frac{1}{a}}\right]_0^p \\
&= \frac{\min\left(a_1, a_2, \ldots, a_k\right)}{p\left(\frac{1}{a}-1\right)} \left[(1-p)^{1-\frac{1}{a}}-1\right].
\end{aligned}$$

(2 marks)

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e) The likelihood function is

$$\begin{aligned}
L\left(a, a_1, \ldots, a_k\right) &= a^n \left[\min\left(a_1, a_2, \ldots, a_k\right)\right]^{na} \left(\prod_{i=1}^n t_i\right)^{-a-1} \left\{\prod_{i=1}^n I\left[t_i > \min\left(a_1, \ldots, a_k\right)\right]\right\} \\
&= a^n \left[\min\left(a_1, a_2, \ldots, a_k\right)\right]^{na} \left(\prod_{i=1}^n t_i\right)^{-a-1} \left\{I\left[\min\left(t_1, \ldots, t_n\right) > \min\left(a_1, \ldots, a_k\right)\right]\right\}.
\end{aligned}$$

As a function of $\min\left(a_1, \ldots, a_k\right)$, it is increasing over $\left(-\infty, \min\left(t_1, \ldots, t_n\right)\right)$. Hence, the maximum likelihood estimator of $a_1, a_2, \ldots, a_k$ are those values satisfying $\min\left(a_1, \ldots, a_k\right) = \min\left(t_1, \ldots, t_n\right)$.

To find the maximum likelihood estimator of a, take the log of the likelihood

$$\log L\left(a, a_1, \ldots, a_k\right) = n \log a + na \log\left[\min\left(a_1, a_2, \ldots, a_k\right)\right] - (a+1) \sum_{i=1}^n \log t_i.$$

The partial derivative with respect to a is

$$\frac{\partial \log L}{\partial a} = \frac{n}{a} + n \log\left[\min\left(a_1, a_2, \ldots, a_k\right)\right] - \sum_{i=1}^n \log t_i.$$

Setting this to zero gives

$$\widehat{a} = n \left\{-n \log\left[\min\left(a_1, a_2, \ldots, a_k\right)\right] + \sum_{i=1}^n \log t_i\right\}^{-1}.$$

This is a maximum likelihood estimator since $\frac{\partial^2 \log L}{\partial a^2} = -\frac{n}{a^2} < 0$.

(8 marks)

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