SOLUTIONS TO MATH68181 EXTREME VALUES AND FINANCIAL RISK EXAM

Solutions to Question A1

a) The marginal cdfs of

$$F_{X,Y}(x,y) = 1 - \exp(-x^\alpha) - \exp(-y^\alpha) + \left[\exp(x^\alpha) + \exp(y^\alpha) - 1\right]^{-1}$$

are

$$F_X(x) = F_{X,Y}(x,\infty) = 1 - \exp(-x^\alpha)$$

and

$$F_Y(y) = F_{X,Y}(\infty,y) = 1 - \exp(-y^\alpha).$$

(3 marks)

UNSEEN

b) First note that $w(F_X) = +\infty$. F_X belongs to the Gumbel max domain of attraction since

$$\begin{aligned}
\lim_{t\to\infty} \frac{1 - F_X(t + x\gamma(t))}{1 - F_X(t)} &= \lim_{t\to\infty} \frac{e^{-(t+x\gamma(t))^\alpha}}{e^{-t^\alpha}} \\
&= \lim_{t\to\infty} e^{t^\alpha - (t+x\gamma(t))^\alpha} \\
&= \lim_{t\to\infty} e^{t^\alpha\left[1-\left(1+\frac{x\gamma(t)}{t}\right)^\alpha\right]} \\
&= \lim_{t\to\infty} e^{t^\alpha\left[1-\left(1+\alpha\frac{x\gamma(t)}{t}\right)\right]} \\
&= \lim_{t\to\infty} e^{-\alpha x t^{\alpha-1}\gamma(t)} \\
&= e^{-x}
\end{aligned}$$

if $\gamma(t) = \alpha^{-1}t^{1-\alpha}$.

(3 marks)

c) First note that $w(F_Y) = +\infty$. F_Y belongs to the Gumbel max domain of attraction since

$$\begin{aligned}
\lim_{t\to\infty} \frac{1 - F_Y(t + y\gamma(t))}{1 - F_Y(t)} &= \lim_{t\to\infty} \frac{e^{-(t+y\gamma(t))^\alpha}}{e^{-t^\alpha}} \\
&= \lim_{t\to\infty} e^{t^\alpha - (t+y\gamma(t))^\alpha} \\
&= \lim_{t\to\infty} e^{t^\alpha\left[1-\left(1+\frac{y\gamma(t)}{t}\right)^\alpha\right]} \\
&= \lim_{t\to\infty} e^{t^\alpha\left[1-\left(1+\alpha\frac{y\gamma(t)}{t}\right)\right]} \\
&= \lim_{t\to\infty} e^{-\alpha y t^{\alpha-1}\gamma(t)} \\
&= e^{-y}
\end{aligned}$$

if $\gamma(t) = \alpha^{-1} t^{1-\alpha}$.

(2 marks)

SEEN

d) Use the rule $a_n = \gamma^{-1}\left(F_X^{-1}(1-n^{-1})\right)$ and $b_n = F_X^{-1}(1-n^{-1})$. Inverting

$$F_X(x) = 1 - \exp(-x^\alpha) = 1 - n^{-1},$$

we obtain

$$F_X^{-1}\left(1-n^{-1}\right) = (\log n)^{1/\alpha}.$$

So,

$$a_n = \frac{1}{\alpha}(\log n)^{\frac{1}{\alpha}-1}$$

we obtain

$$b_n = (\log n)^{1/\alpha}$$

(3 marks)

UNSEEN

e) Use the rule $c_n = \gamma^{-1}\left(F_Y^{-1}(1-n^{-1})\right)$ and $d_n = F_Y^{-1}(1-n^{-1})$. Inverting

$$F_Y(y) = 1 - \exp(-y^\alpha) = 1 - n^{-1},$$

we obtain

$$F_Y^{-1}\left(1-n^{-1}\right) = (\log n)^{1/\alpha}.$$

So,

$$c_n = \frac{1}{\alpha}(\log n)^{\frac{1}{\alpha}-1}$$

we obtain

$$d_n = (\log n)^{1/\alpha}$$

(2 marks)

UNSEEN

g) The limiting cdf is

$$
\begin{aligned}
& \lim_{n\to\infty} F_{X,Y}\left(a_n x + b_n, c_n y + d_n\right) \\
= & \lim_{n\to\infty} \left\{ 1 - \exp\left\{ -\left[\frac{1}{\alpha} (\log n)^{\frac{1}{\alpha}-1} x + (\log n)^{\frac{1}{\alpha}} \right]^{\alpha} \right\} \right. \\
& \quad - \exp\left\{ -\left[\frac{1}{\alpha} (\log n)^{\frac{1}{\alpha}-1} y + (\log n)^{\frac{1}{\alpha}} \right]^{\alpha} \right\} \\
& \quad \left. + \left[\exp\left\{ \left[\frac{1}{\alpha} (\log n)^{\frac{1}{\alpha}-1} x + (\log n)^{\frac{1}{\alpha}} \right]^{\alpha} \right\} + \exp\left\{ \left[\frac{1}{\alpha} (\log n)^{\frac{1}{\alpha}-1} y + (\log n)^{\frac{1}{\alpha}} \right]^{\alpha} \right\} - 1 \right]^{-1} \right\}^{n} \\
= & \lim_{n\to\infty} \left\{ 1 - \exp\left\{ -\log n \left[\frac{1}{\alpha} (\log n)^{-1} x + 1 \right]^{\alpha} \right\} \right. \\
& \quad - \exp\left\{ -\log n \left[\frac{1}{\alpha} (\log n)^{-1} y + 1 \right]^{\alpha} \right\} \\
& \quad \left. + \left[\exp\left\{ \log n \left[\frac{1}{\alpha} (\log n)^{-1} x + 1 \right]^{\alpha} \right\} + \exp\left\{ \log n \left[\frac{1}{\alpha} (\log n)^{-1} y + 1 \right]^{\alpha} \right\} - 1 \right]^{-1} \right\}^{n} \\
= & \lim_{n\to\infty} \left\{ 1 - \exp\left\{ -\log n \left[1 + (\log n)^{-1} x + \cdots \right] \right\} \right. \\
& \quad - \exp\left\{ -\log n \left[1 + (\log n)^{-1} y + \cdots \right] \right\} \\
& \quad \left. + \left[\exp\left\{ \log n \left[1 + (\log n)^{-1} x + \cdots \right] \right\} + \exp\left\{ \log n \left[1 + (\log n)^{-1} y + \cdots \right] \right\} - 1 \right]^{-1} \right\}^{n} \\
= & \lim_{n\to\infty} \left\{ 1 - \exp\left\{ -\log n - x \right\} \right. \\
& \quad - \exp\left\{ -\log n - y \right\} \\
& \quad \left. + \left[\exp\left\{ \log n + x \right\} + \exp\left\{ \log n + y \right\} - 1 \right]^{-1} \right\}^{n} \\
= & \lim_{n\to\infty} \left\{ 1 - \frac{\exp(-x)}{n} - \frac{\exp(-y)}{n} + \left[n \exp(x) + n \exp(y) - 1 \right]^{-1} \right\}^{n} \\
= & \lim_{n\to\infty} \left\{ 1 - \frac{\exp(-x)}{n} - \frac{\exp(-y)}{n} + \frac{1}{n} \left[\exp(x) + \exp(y) - \frac{1}{n} \right]^{-1} \right\}^{n} \\
= & \lim_{n\to\infty} \exp\left\{ -\exp(-x) - \exp(-y) + \left[\exp(x) + \exp(y) - \frac{1}{n} \right]^{-1} \right\} \\
= & \exp\left\{ -\exp(-x) - \exp(-y) + \left[\exp(x) + \exp(y) \right]^{-1} \right\}.
\end{aligned}
$$

(5 marks)

UNSEEN

h) No, the extremes are not completely independent since

$$\begin{aligned}
\lim_{n\to\infty} F_{X,Y}\left(a_n x + b_n, c_n y + d_n\right) &= \exp\left\{-\exp\left(-x\right) - \exp\left(-y\right) + \left[\exp\left(x\right) + \exp\left(y\right)\right]^{-1}\right\} \\
&\neq \exp\left\{-\exp\left(-x\right) - \exp\left(-y\right)\right\} \\
&= \lim_{n\to\infty} F_X\left(a_n x + b_n\right) \lim_{n\to\infty} F_Y\left(c_n y + d_n\right).
\end{aligned}$$

(2 marks)

UNSEEN

Solutions to Question A2

$C(u_1, u_2)$ is a valid copula if

$$C(u, 0) = 0,$$

$$C(0, u) = 0,$$

$$C(1, u) = u,$$

$$C(u, 1) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) \geq 0.$$

(4 marks)

SEEN

a) for the copula defined by $C(u_1, u_2) = \alpha C_1(u_1, u_2) + (1-\alpha) C_2(u_1, u_2)$, we have

$$C(u, 0) = \alpha C_1(u, 0) + (1-\alpha) C_2(u, 0) = \alpha \cdot 0 + (1-\alpha) \cdot 0 = 0,$$

$$C(0, u) = \alpha C_1(0, u) + (1-\alpha) C_2(0, u) = \alpha \cdot 0 + (1-\alpha) \cdot 0 = 0,$$

$$C(1, u) = \alpha C_1(1, u) + (1-\alpha) C_2(1, u) = \alpha \cdot 1 + (1-\alpha) \cdot 1 = 1,$$

$$C(u, 1) = \alpha C_1(u, 1) + (1-\alpha) C_2(u, 1) = \alpha \cdot 1 + (1-\alpha) \cdot 1 = 1,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \alpha \frac{\partial}{\partial u_1} C_1(u_1, u_2) + (1-\alpha) \frac{\partial}{\partial u_1} C(u_1, u_2) \geq 0 + 0 = 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \alpha \frac{\partial}{\partial u_2} C_1(u_1, u_2) + (1-\alpha) \frac{\partial}{\partial u_2} C(u_1, u_2) \geq 0 + 0 = 0.$$

(4 marks)

UNSEEN

b) for the copula defined by $C(u_1, u_2) = \frac{u_1 u_2}{u_1+u_2-u_1 u_2}$, we have

$$C(u, 0) = \frac{u \cdot 0}{u + 0 - u \cdot 0} = 0,$$

$$C(0, u) = \frac{0 \cdot u}{0 + u - 0 \cdot u} = 0,$$

$$C(1, u) = \frac{1 \cdot u}{1 + u - 1 \cdot u} = u,$$

$$C(u, 1) = \frac{u \cdot 1}{u + 1 - u \cdot 1} = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \frac{\partial}{\partial u_1} \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2} = \frac{(u_1 + u_2 - u_1 u_2) u_1 - u_1 u_2 (1 - u_1)}{(u_1 + u_2 - u_1 u_2)^2} = \frac{u_1^2}{(u_1 + u_2 - u_1 u_2)^2} \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \frac{\partial}{\partial u_2} \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2} = \frac{(u_1 + u_2 - u_1 u_2) u_2 - u_1 u_2 (1 - u_2)}{(u_1 + u_2 - u_1 u_2)^2} = \frac{u_2^2}{(u_1 + u_2 - u_1 u_2)^2} \geq 0.$$

(4 marks)

UNSEEN

c) for the copula defined by $C(u_1, u_2) = u_1 u_2 + \alpha u_1 u_2 (1 - u_1)(1 - u_2)$, we have

$$C(u, 0) = u \cdot 0 + \alpha u \cdot 0 \cdot (1 - u)(1 - 0) = 0,$$

$$C(0, u) = 0 \cdot u + \alpha 0 \cdot u \cdot (1 - 0)(1 - u) = 0,$$

$$C(1, u) = 1 \cdot u + \alpha 1 \cdot u \cdot (1 - 1)(1 - u) = u,$$

$$C(u, 1) = u \cdot 1 + \alpha u \cdot 1 \cdot (1 - u)(1 - 1) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = u_2 + \alpha u_2 (1 - 2u_1)(1 - u_2) = u_2 [1 + \alpha (1 - 2u_1)(1 - u_2)] \geq 0$$

since $-1 \leq \alpha (1-2u_1)(1-u_2) \leq 1$ and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = u_1 + \alpha u_1 (1-2u_2)(1-u_1) = u_1 [1 + \alpha (1-2u_2)(1-u_1)] \geq 0$$

since $-1 \leq \alpha (1-2u_2)(1-u_1) \leq 1$.

(4 marks)

UNSEEN

d) for the copula defined by $C(u_1, u_2) = \begin{cases} \max(u_1 + u_2 - 1, t), & \text{if } t \leq u_1 \leq 1, t \leq u_2 \leq 1, \\ \min(u_1, u_2), & \text{otherwise,} \end{cases}$
we have

$$C(u, 0) = \min(u, 0) = 0,$$

$$C(0, u) = \min(0, u) = 0,$$

$$C(1, u) = \begin{cases} \max(1 + u - 1, t), & \text{if } t \leq u \leq 1, \\ \min(1, u), & \text{otherwise} \end{cases} = u,$$

$$C(u, 1) = \begin{cases} \max(u + 1 - 1, t), & \text{if } t \leq u \leq 1, \\ \min(u, 1), & \text{otherwise} \end{cases} = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \begin{cases} \frac{\partial}{\partial u_1} \max(u_1 + u_2 - 1, t), & \text{if } t \leq u_1 \leq 1, t \leq u_2 \leq 1, \\ \frac{\partial}{\partial u_1} \min(u_1, u_2), & \text{otherwise} \end{cases} \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \begin{cases} \frac{\partial}{\partial u_2} \max(u_1 + u_2 - 1, t), & \text{if } t \leq u_1 \leq 1, t \leq u_2 \leq 1, \\ \frac{\partial}{\partial u_2} \min(u_1, u_2), & \text{otherwise} \end{cases} \geq 0.$$

(4 marks)

UNSEEN

Solutions to Question A3

a) We can write

$$\overline{F}(x,y) = \exp\left\{ -(x+y) \sum_{i=1}^{k} \alpha_i A_i \left(\frac{y}{x+y} \right) \right\}$$

for $x > 0$, $y > 0$ and $\alpha_i \geq 0$ sum to one. This is in the form of

$$\overline{F}(x,y) = \exp\left[-(x+y) A \left(\frac{y}{x+y} \right) \right]$$

with

$$A(w) = \sum_{i=1}^{k} \alpha_i A_i(w).$$

We now check the conditions for $A(\cdot)$. Clearly,

$$A(0) = \sum_{i=1}^{k} \alpha_i A_i(0) = \sum_{i=1}^{k} \alpha_i \cdot 1 = 1$$

and

$$A(1) = \sum_{i=1}^{k} \alpha_i A_i(1) = \sum_{i=1}^{k} \alpha_i \cdot 1 = 1.$$

Also $A(t) \geq 0$ since $A_i(w) \geq 0$ for all w and every k.

Also since each $A_i(w) \leq 1$,

$$A(w) = \sum_{i=1}^{k} \alpha_i A_i(w) \leq \sum_{i=1}^{k} \alpha_i \cdot 1 = 1.$$

Also since each $A_i(w) \geq \max(w, 1-w)$,

$$A(w) = \sum_{i=1}^{k} \alpha_i A_i(w) \geq \sum_{i=1}^{k} \alpha_i \cdot \max(w, 1-w) = \max(w, 1-w).$$

Also since each $A_i(w)$ is convex,

$$A''(w) = \sum_{i=1}^{k} \alpha_i A_i''(w) \geq 0.$$

(7 marks)

UNSEEN

b) Note that

$$\overline{F}(x,0)=\exp\left\{-(x+0)\sum_{i=1}^{k}\alpha_i A_i\left(0\right)\right\}=\exp(-x)$$

and

$$\overline{F}(0,y)=\exp\left\{-(0+y)\sum_{i=1}^{k}\alpha_i A_i\left(0\right)\right\}=\exp(-y).$$

So, the joint cdf is

$$F(x,y)=1-\exp(-x)-\exp(-y)+\exp\left\{-(x+y)\sum_{i=1}^{k}\alpha_i A_i\left(\frac{y}{x+y}\right)\right\}.$$

(1 marks)

UNSEEN

c) the derivative of joint cdf with respect to x is

$$\frac{\partial F(x,y)}{\partial x}=\exp(-x)+\overline{F}(x,y)\left\{-\sum_{i=1}^{k}\alpha_i A_i\left(\frac{y}{x+y}\right)+\frac{y}{x+y}\sum_{i=1}^{k}\alpha_i A_i^{'}\left(\frac{y}{x+y}\right)\right\},$$

so the conditional cdf of Y given $X=x$ is

$$F(y\mid x)=1+\exp(x)\overline{F}(x,y)\left\{-\sum_{i=1}^{k}\alpha_i A_i\left(\frac{y}{x+y}\right)+\frac{y}{x+y}\sum_{i=1}^{k}\alpha_i A_i^{'}\left(\frac{y}{x+y}\right)\right\}.$$

(4 marks)

UNSEEN

d) the derivative of joint cdf with respect to y is

$$\frac{\partial F(x,y)}{\partial y}=\exp(-y)+\overline{F}(x,y)\left\{-\sum_{i=1}^{k}\alpha_i A_i\left(\frac{y}{x+y}\right)-\frac{x}{x+y}\sum_{i=1}^{k}\alpha_i A_i^{'}\left(\frac{y}{x+y}\right)\right\},$$

so the conditional cdf of Y given $X = x$ is

$$F(x \mid y) = 1 + \exp(y)\overline{F}(x,y)\left\{-\sum_{i=1}^{k}\alpha_i A_i\left(\frac{y}{x+y}\right) - \frac{x}{x+y}\sum_{i=1}^{k}\alpha_i A_i^{'}\left(\frac{y}{x+y}\right)\right\}.$$

(4 marks)

UNSEEN

e) the derivative of joint cdf with respect to x and y is

$$\begin{aligned} f(x,y) &= \overline{F}(x,y)\left\{-\sum_{i=1}^{k}\alpha_i A_i\left(\frac{y}{x+y}\right) + \frac{y}{x+y}\sum_{i=1}^{k}\alpha_i A_i^{'}\left(\frac{y}{x+y}\right)\right\} \\ &\quad \cdot\left\{-\sum_{i=1}^{k}\alpha_i A_i\left(\frac{y}{x+y}\right) - \frac{x}{x+y}\sum_{i=1}^{k}\alpha_i A_i^{'}\left(\frac{y}{x+y}\right)\right\} \\ &\quad +\overline{F}(x,y)\frac{xy}{(x+y)^2}\sum_{i=1}^{k}\alpha_i A_i^{''}\left(\frac{y}{x+y}\right). \end{aligned}$$

(4 marks)

UNSEEN

Solutions to Question B1

a) The cumulative distribution function of T conditional on $N = n$ is

$$\begin{aligned}
\Pr(T \le t \mid N = n) &= \Pr(\max(X_1, \ldots, X_N) \le t \mid N = n) \\
&= \Pr(\max(X_1, \ldots, X_n) \le t \mid N = n) \\
&= \Pr(X_1 \le t, \ldots, X_n \le t) \\
&= \Pr(X_1 \le t) \cdots \Pr(X_n \le t) \\
&= \frac{t+a}{2a} \cdots \frac{t+a}{2a} \\
&= \left(\frac{t+a}{2a}\right)^n .
\end{aligned}$$

(4 marks)

UNSEEN

b) The unconditional cumulative distribution function of T is

$$\begin{aligned}
\Pr(T \le t) &= \sum_{n=0}^{\infty} \Pr(T \le t \mid N = n) \frac{\theta^n \exp(-\theta)}{n!} \\
&= \sum_{n=0}^{\infty} \left(\frac{t+a}{2a}\right)^n \frac{\theta^n \exp(-\theta)}{n!} \\
&= \exp(-\theta) \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\theta(t+a)}{2a}\right)^n \\
&= \exp\left(-\theta + \frac{\theta(t+a)}{2a}\right) \\
&= \exp\left(\frac{\theta(t-a)}{2a}\right) .
\end{aligned}$$

(4 marks)

UNSEEN

c) The unconditional probability density function of T is

$$f_T(t) = \frac{\theta}{2a} \exp\left(\frac{\theta(t-a)}{2a}\right) .$$

(1 marks)

UNSEEN

d) The moment generating function of T is

$$\begin{aligned} M_T(s) &= \frac{\theta}{2a} \exp\left(-\frac{\theta}{2}\right) \int_{-a}^{a} \exp\left(st + \frac{\theta t}{2a}\right) dt \\ &= \frac{\theta}{2a} \exp\left(-\frac{\theta}{2}\right) \left(s + \frac{\theta}{2a}\right)^{-1} \left[\exp\left(st + \frac{\theta t}{2a}\right)\right]_{-a}^{a} \\ &= \frac{\theta}{2a} \exp\left(-\frac{\theta}{2}\right) \left(s + \frac{\theta}{2a}\right)^{-1} \left[\exp\left(sa + \frac{\theta}{2}\right) - \exp\left(-sa - \frac{\theta}{2}\right)\right]. \end{aligned}$$

(4 marks)

UNSEEN

e) Setting

$$\exp\left(\frac{\theta(t-a)}{2a}\right). = p$$

and solving for t, we obtain value at risk of T as

$$\mathrm{VaR}_p(T) = a + \frac{2a}{\theta} \log p.$$

(3 marks)

UNSEEN

f) The expected shortfall T is

$$\begin{aligned} \mathrm{ES}_p(T) &= a + \frac{2a}{\theta p} \int_0^p \log x dx \\ &= a + \frac{2a}{\theta p} \left\{[t \log t]_0^p - p\right\} \\ &= a + \frac{2a}{\theta p} \left\{p \log p - p\right\} \\ &= a + \frac{2a}{\theta} \left\{\log p - 1\right\}. \end{aligned}$$

(4 marks)

UNSEEN

Solutions to Question B2

If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cdf of a normalized version of M_n converges to G, i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n\left(a_n x + b_n\right) \to G(x) \tag{1}$$

as $n \to \infty$ then G must be of the same type as (cdfs G and G^* are of the same type if $G^*(x) = G(ax+b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$\begin{aligned}
I &: \Lambda(x) = \exp\left\{-\exp(-x)\right\}, \qquad x \in \Re; \\
II &: \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\left\{-x^{-\alpha}\right\} & \text{if } x \geq 0 \end{cases} \\
&\quad \text{for some } \alpha > 0; \\
III &: \Psi_\alpha(x) = \begin{cases} \exp\left\{-(-x)^{\alpha}\right\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \\
&\quad \text{for some } \alpha > 0.
\end{aligned}$$

(4 marks)

SEEN

The necessary and sufficient conditions for the three extreme value distributions are:

$$\begin{aligned}
I &: \exists \gamma(t) > 0 \text{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F\left(t + x\gamma(t)\right)}{1 - F(t)} = \exp(-x), \qquad x \in \Re, \\
II &: w(F) = \infty \text{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \qquad x > 0, \\
III &: w(F) < \infty \text{ and } \lim_{t \downarrow 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} = x^{\alpha}, \qquad x > 0.
\end{aligned}$$

(4 marks)

SEEN

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function say $h(t)$ such that

$$\lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} = e^{-x}$$

for every $x \in (-\infty, \infty)$. But

$$
\begin{aligned}
\lim_{t \to w(F)} \frac{1 - F\left(t + xh(t)\right)}{1 - F(t)} &= \lim_{t \to w(F)} \frac{\left(1 + xh^{'}(t)\right) f\left(t + xh(t)\right)}{f(t)} \\
&= \lim_{t \to w(F)} \frac{\left(1 + xh^{'}(t)\right) g\left(t + xh(t)\right) G^{a-1}\left(t + xh(t)\right) \left[1 - G\left(t + xh(t)\right)\right]^{b-1}}{g\left(t\right) G^{a-1}\left(t\right) \left[1 - G\left(t\right)\right]^{b-1}} \\
&= \lim_{t \to w(F)} \frac{\left(1 + xh^{'}(t)\right) g\left(t + xh(t)\right) \left[1 - G\left(t + xh(t)\right)\right]^{b-1}}{g\left(t\right) \left[1 - G\left(t\right)\right]^{b-1}} \\
&= \lim_{t \to w(F)} \frac{\left(1 + xh^{'}(t)\right) g\left(t + xh(t)\right)}{g\left(t\right)} \frac{\left[1 - G\left(t + xh(t)\right)\right]^{b-1}}{\left[1 - G\left(t\right)\right]^{b-1}} \\
&= \lim_{t \to w(F)} \frac{1 - G\left(t + xh(t)\right)}{1 - G\left(t\right)} \frac{\left[1 - G\left(t + xh(t)\right)\right]^{b-1}}{\left[1 - G\left(t\right)\right]^{b-1}} \\
&= \lim_{t \to w(F)} \left[\frac{1 - G\left(t + xh(t)\right)}{1 - G\left(t\right)}\right]^{b} \\
&= \exp(-bx)
\end{aligned}
$$

for every $x \in (-\infty, \infty)$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$
\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left[-\exp(-bx)\right]
$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$
\lim_{t \to \infty} \frac{1 - G\left(tx\right)}{1 - G(t)} = x^{-\beta}
$$

for every $x > 0$. But

$$
\begin{aligned}
\lim_{t \to w(F)} \frac{1 - F(tx)}{1 - F(t)} &= \lim_{t \to w(F)} \frac{x f(tx)}{f(t)} \\
&= \lim_{t \to w(F)} \frac{x g(tx) G^{a-1}(tx) \left[1 - G(tx)\right]^{b-1}}{g(t) G^{a-1}(t) \left[1 - G(t)\right]^{b-1}} \\
&= \lim_{t \to w(F)} \frac{x g(tx) \left[1 - G(tx)\right]^{b-1}}{g(t) \left[1 - G(t)\right]^{b-1}} \\
&= \lim_{t \to w(F)} \frac{x g(tx)}{g(t)} \frac{\left[1 - G(tx)\right]^{b-1}}{\left[1 - G(t)\right]^{b-1}} \\
&= \lim_{t \to w(F)} \frac{1 - G(tx)}{1 - G(t)} \frac{\left[1 - G(tx)\right]^{b-1}}{\left[1 - G(t)\right]^{b-1}} \\
&= \lim_{t \to w(F)} \left[\frac{1 - G(t + x h(t))}{1 - G(t)}\right]^{b} \\
&= x^{-b\beta}
\end{aligned}
$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$
\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left(-x^{-b\beta}\right)
$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$
\lim_{t \to 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} = x^{\beta}
$$

for every $x > 0$. But

$$
\begin{aligned}
\lim_{t \downarrow 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F(w(F) - t)} &= \lim_{t \downarrow 0} \frac{x f\left(w(F) - tx\right)}{f(w(F) - t)} \\
&= \lim_{t \downarrow 0} \frac{x g\left(w(F) - tx\right) G^{a-1}\left(w(F) - tx\right) \left[1 - G\left(w(F) - tx\right)\right]^{b-1}}{g\left(t\right) G^{a-1}\left(w(F) - t\right) \left[1 - G\left(w(F) - t\right)\right]^{b-1}} \\
&= \lim_{t \downarrow 0} \frac{x g\left(w(F) - tx\right) \left[1 - G\left(w(F) - tx\right)\right]^{b-1}}{g\left(w(F) - t\right) \left[1 - G\left(w(F) - t\right)\right]^{b-1}} \\
&= \lim_{t \downarrow 0} \frac{x g\left(w(F) - tx\right)}{g\left(w(F) - t\right)} \frac{\left[1 - G\left(w(F) - tx\right)\right]^{b-1}}{\left[1 - G\left(w(F) - t\right)\right]^{b-1}} \\
&= \lim_{t \downarrow 0} \frac{1 - G\left(w(F) - tx\right)}{1 - G(w(F) - t)} \frac{\left[1 - G\left(w(F) - tx\right)\right]^{b-1}}{\left[1 - G\left(w(F) - t\right)\right]^{b-1}} \\
&= \lim_{t \downarrow 0} \left[\frac{1 - G\left(w(F) - tx\right)}{1 - G(w(F) - t)}\right]^{b} \\
&= x^{b\beta}
\end{aligned}
$$

for every $x < 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$
\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left(-(-x)^{b\beta}\right)
$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

UNSEEN

Solutions to Question B3

a) Note that $w(F) = N$ and

$$\begin{aligned}
\frac{\Pr(X = w(F))}{1 - F(w(F) - 1)} &= \frac{\Pr(X = N)}{1 - F(N - 1)} \\
&= \frac{\Pr(X = N)}{1 - \Pr(X = 1) - \Pr(X = 2) - \cdots - \Pr(X = N - 1)} \\
&= \frac{\frac{1}{N}}{1 - \frac{1}{N} - \frac{1}{N} - \cdots - \frac{1}{N}} \\
&= \frac{\frac{1}{N}}{1 - \frac{N-1}{N}} \\
&= 1.
\end{aligned}$$

Hence, there can be no non-degenerate limit.

(4 marks)

UNSEEN

b) Note that $w(F) = n$ and

$$\begin{aligned}
\frac{\Pr(X = w(F))}{1 - F(w(F) - 1)} &= \frac{\Pr(X = n)}{1 - F(n - 1)} \\
&= \frac{\Pr(X = n)}{1 - \Pr(X \leq n - 1)} \\
&= \frac{\Pr(X = n)}{\Pr(X = n)} \\
&= 1.
\end{aligned}$$

Hence, there can be no non-degenerate limit.

c) Note that $w(F) = b$. The corresponding cdf is

$$F(x) = \frac{\log x - \log a}{\log b - \log a}.$$

Note that

$$\begin{aligned}
\lim_{t\downarrow 0}\frac{1-F\left(w(F)-tx\right)}{1-F\left(w(F)-t\right)} &= \lim_{t\downarrow 0}\frac{1-F\left(b-tx\right)}{1-F\left(b-t\right)} \\
&= \lim_{t\downarrow 0}\frac{1-\frac{\log(b-tx)-\log a}{\log b-\log a}}{1-\frac{\log(b-t)-\log a}{\log b-\log a}} \\
&= \lim_{t\downarrow 0}\frac{\log b-\log(b-tx)}{\log b-\log(b-t)} \\
&= \lim_{t\downarrow 0}\frac{\frac{x}{b-tx}}{\frac{1}{b-t}} \\
&= x.
\end{aligned}$$

So, $F(x)$ belongs to the Weibull domain of attraction.

(4 marks)

UNSEEN

d) Note that $w(F)=\infty$. Then

$$\begin{aligned}
\lim_{t\to\infty}\frac{1-F\left(t+xg(t)\right)}{1-F(t)} &= \lim_{t\to\infty}\frac{1-\left\{1-\exp\left[-\left(t+xg(t)\right)^2\right]\right\}^a}{1-\left\{1-\exp\left[-t^2\right]\right\}^a} \\
&= \lim_{t\to\infty}\frac{1-\left\{1-a\exp\left[-\left(t+xg(t)\right)^2\right]\right\}}{1-\left\{1-a\exp\left[-t^2\right]\right\}} \\
&= \lim_{t\to\infty}\frac{\exp\left[-\left(t+xg(t)\right)^2\right]}{\exp\left[-t^2\right]} \\
&= \lim_{t\to\infty}\exp\left[-2txg(t)-x^2g^2(t)\right] \\
&= \exp(-x)
\end{aligned}$$

if $g(t)=1/(2t)$. So, $F(x)=\left[1-\exp\left(-x^2\right)\right]^a$ belongs to the Gumbel domain of attraction.

(4 marks)

UNSEEN

e) Note that $w(F) = \infty$. Then

$$\begin{aligned}
\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} &= \lim_{t \uparrow \infty} \frac{1 - 1 + \left[1 - \exp\left(-(tx)^{-1}\right)\right]^a}{1 - 1 + \left[1 - \exp\left(-t^{-1}\right)\right]^a} \\
&= \lim_{t \uparrow \infty} \frac{\left[1 - \exp\left(-(tx)^{-1}\right)\right]^a}{\left[1 - \exp\left(-t^{-1}\right)\right]^a} \\
&= \lim_{t \uparrow \infty} \left[\frac{1 - \exp\left(-(tx)^{-1}\right)}{1 - \exp\left(-t^{-1}\right)}\right]^a \\
&= \lim_{t \uparrow \infty} \left[\frac{1 - 1 + (tx)^{-1}}{1 - 1 + t^{-1}}\right]^a \\
&= x^{-a}.
\end{aligned}$$

. So, the cdf $F(x) = 1 - \left[1 - \exp\left(-x^{-1}\right)\right]^a$ belongs to the Fréchet domain of attraction.

(4 marks)

UNSEEN

Solutions to Question B4

(a) If X is an absolutely continuous random variable with cdf $F(\cdot)$ then

$$\mathrm{VaR}_p(X) = F^{-1}(p)$$

and

$$\mathrm{ES}_p(X) = \frac{1}{p}\int_0^p F^{-1}(v)dv.$$

(2 marks)

SEEN

(b) (i) The corresponding cdf is

$$F(x) = \int_{-a}^{x} \frac{3y^2}{2a^3}dy = \frac{3}{2a^3}\left[\frac{y^3}{3}\right]_{-a}^{x} = \frac{3}{2a^3}\left[\frac{x^3+a^3}{3}\right] = \frac{x^3+a^3}{2a^3}$$

for $-a < x < a$;

(2 marks)

UNSEEN

(b) (ii) Inverting

$$F(x) = \frac{x^3+a^3}{2a^3} = p,$$

we obtain $\mathrm{VaR}_p(X) = (2p-1)^{1/3}a$.

(1 marks)

UNSEEN

(b) (iii) The expected shortfall is

$$\mathrm{ES}_p(X) = \frac{1}{p}\int_0^p (2v-1)^{1/3}dv = \frac{1}{p}\left[\frac{3(2v-1)^{4/3}}{8}\right]_0^p = \frac{3a}{8p}\left[(2p-1)^{4/3}-1\right].$$

(2 marks)

UNSEEN

c) (i) The likelihood function of a is

$$\begin{aligned}
L(a) &= \frac{3^n}{2^n a^{3n}} \prod_{i=1}^{n} \left[X_i^2 I\left\{-a < X_i < a\right\} \right] \\
&= \frac{3^n}{2^n a^{3n}} \left(\prod_{i=1}^{n} X_i \right)^2 \prod_{i=1}^{n} I\left\{-a < X_i < a\right\} \\
&= \frac{3^n}{2^n a^{3n}} \left(\prod_{i=1}^{n} X_i \right)^2 I\left\{\max\left(X_1, \ldots, X_n\right) < a, \min\left(X_1, \ldots, X_n\right) > -a\right\} \\
&= \frac{3^n}{2^n a^{3n}} \left(\prod_{i=1}^{n} X_i \right)^2 I\left\{a > \max\left(X_1, \ldots, X_n\right), a > -\min\left(X_1, \ldots, X_n\right)\right\} \\
&= \frac{3^n}{2^n a^{3n}} \left(\prod_{i=1}^{n} X_i \right)^2 I\left\{a > \max\left[\max\left(X_1, \ldots, X_n\right), -\min\left(X_1, \ldots, X_n\right)\right]\right\}.
\end{aligned}$$

(3 marks)

UNSEEN

c) (ii) Note that $L(a)$ is a decreasing function over a and $a > \max\left[\max\left(X_1, \ldots, X_n\right), -\min\left(X_1, \ldots, X_n\right)\right]$. Hence, the mle of a is $\max\left[\max\left(X_1, \ldots, X_n\right), -\min\left(X_1, \ldots, X_n\right)\right]$.

(1 marks)

UNSEEN

c) (iii) The mles of VaR and ES are $\widehat{\mathrm{VaR}}_p(X) = (2p-1)^{1/3}\widehat{a}$ and

$$\widehat{\mathrm{ES}}_p(X) = \frac{3\widehat{a}}{8p}\left[(2p-1)^{4/3} - 1\right],$$

where $\widehat{a} = \max\left[\max\left(X_1, \ldots, X_n\right), -\min\left(X_1, \ldots, X_n\right)\right]$.

(2 marks)

UNSEEN

c (iv) Let $Z = \max\left[\max\left(X_1, \ldots, X_n\right), -\min\left(X_1, \ldots, X_n\right)\right]$. The cdf of Z is

$$\begin{aligned}
F_Z(z) &= \Pr\left(\max\left[\max\left(X_1, \ldots, X_n\right), -\min\left(X_1, \ldots, X_n\right)\right] \leq z\right) \\
&= \Pr\left(\max\left(X_1, \ldots, X_n\right) \leq z, -\min\left(X_1, \ldots, X_n\right) \leq z\right) \\
&= \Pr\left(\max\left(X_1, \ldots, X_n\right) \leq z, \min\left(X_1, \ldots, X_n\right) \geq -z\right) \\
&= \Pr\left(X_1 \leq z, \ldots, X_n \leq z, X_1 \geq -z, \ldots, X_n \geq -z\right) \\
&= \Pr\left(-z \leq X_1 \leq z, \ldots, -z \leq X_n \leq z\right) \\
&= \Pr^n\left(-z \leq X \leq z\right) \\
&= \left[\Pr\left(X \leq z\right) - \Pr\left(X \leq -z\right)\right]^n \\
&= \left[\frac{z^3}{a^3}\right]^n
\end{aligned}$$

for $z > 0$. The corresponding pdf is

$$f_Z(z) = 3na^{-3n}z^{3n-1}$$

for $z > 0$.

(3 marks)

UNSEEN

c (v) The expected value of Z is

$$E(Z) = 3na^{-3n}\int_0^a z^{3n}dz = 3na^{-3n}\left[\frac{z^{3n+1}}{3n+1}\right]_0^a dz = 3na^{-3n}\frac{a^{3n+1}}{3n+1} = \frac{3na}{3n+1}.$$

Hence, $\widehat{a}$ is biased for a.

(2 marks)

UNSEEN

c (vi) Since

$$\text{Bias}\left[\widehat{\text{VaR}}_p(X)\right] = (2p-1)^{1/3}\text{Bias}\left[\widehat{a}\right],$$

we see that $\widehat{\text{VaR}}_p(X)$ is biased for $\text{VaR}_p(X)$.

(1 marks)

UNSEEN

c (vii) Since

$$\text{Bias}\left[\widehat{\text{ES}}_p(X)\right] = \frac{3}{8p}\left[(2p-1)^{4/3} - 1\right]\text{Bias}\left[\widehat{a}\right],$$

we see that $\widehat{\mathrm{ES}}_p(X)$ is biased for $\mathrm{ES}_p(X)$.

(1 marks)

UNSEEN

Solutions to Question B5

a) Note that

$$\frac{\partial \overline{F}}{\partial x_1} = -\frac{\alpha}{\theta} \left(1 + \frac{1}{\theta} \sum_{i=1}^{k} x_i \right)^{-\alpha-1},$$

$$\frac{\partial^2 \overline{F}}{\partial x_1 \partial x_2} = \frac{\alpha(\alpha+1)}{\theta^2} \left(1 + \frac{1}{\theta} \sum_{i=1}^{k} x_i \right)^{-\alpha-2},$$

$$\frac{\partial^3 \overline{F}}{\partial x_1 \partial x_2 \partial x_3} = -\frac{\alpha(\alpha+1)(\alpha+3)}{\theta^3} \left(1 + \frac{1}{\theta} \sum_{i=1}^{k} x_i \right)^{-\alpha-3},$$

and so on. In general,

$$f\left(x_1, x_2, \ldots, x_k\right) = \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\theta^k} \left(1 + \frac{1}{\theta} \sum_{i=1}^{k} x_i \right)^{-\alpha-k},$$

as required.

(4 marks)

UNSEEN

b) The pdf of S is

$$\begin{aligned}
f_S(s) &= \int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{k-1}} f\left(x_1, x_2, \ldots, t - x_1 - x_2 - \cdots - x_{k-1}\right) dx_k \cdots dx_2 dx_1 \\
&= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\theta^k} \int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{k-1}} \left(1 + \frac{1}{\theta} \sum_{i=1}^{k} x_i \right)^{-\alpha-k} dx_k \cdots dx_2 dx_1 \\
&= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\theta^k} \int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{k-1}} \left(1 + \frac{s}{\theta} \right)^{-\alpha-k} dx_k \cdots dx_2 dx_1 \\
&= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\theta^k} \left(1 + \frac{s}{\theta} \right)^{-\alpha-k} \int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{k-1}} dx_k \cdots dx_2 dx_1 \\
&= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!\theta^k} s^k \left(1 + \frac{s}{\theta} \right)^{-\alpha-k},
\end{aligned}$$

as required.

(4 marks)

UNSEEN

c) The cdf of S is

$$\begin{aligned}
F_S(s) &= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!\theta^k}\int_0^s x^k\left(1+\frac{x}{\theta}\right)^{-\alpha-k}dx \\
&= \frac{\theta\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!}\int_1^{\theta/(s+\theta)} y^{\alpha-2}(1-y)^k dy \\
&= \frac{\theta\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!}\int_0^{s/(s+\theta)} (1-y)^{\alpha-2}y^k dy \\
&= \frac{\theta\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!}B_{s/(s+\theta)}(k+1,\alpha-1),
\end{aligned}$$

as required.

(4 marks)

UNSEEN

d) The nth moment of S is

$$\begin{aligned}
E\left(S^n\right) &= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!\theta^k}\int_0^\infty x^{n+k}\left(1+\frac{x}{\theta}\right)^{-\alpha-k}dx \\
&= \frac{\theta^{n+1}\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!}\int_0^1 (1-y)^{n+k}y^{\alpha-n-2}dy \\
&= \frac{\theta^{n+1}\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!}B\left(\alpha-n-1,n+k+1\right).
\end{aligned}$$

(4 marks)

UNSEEN

e) The joint likelihood function is

$$L\left(\alpha,\theta\right) = \frac{\alpha^n(\alpha+1)^n\cdots(\alpha+k-1)^n}{\left(k!\right)^n\theta^{kn}}\left(\prod_{i=1}^n s_i\right)^k\left[\prod_{i=1}^n\left(1+\frac{s_i}{\theta}\right)\right]^{-\alpha-k}.$$

Its log is The joint likelihood function is

$$\log L\left(\alpha,\theta\right) = \log\left[\frac{\alpha^n(\alpha+1)^n\cdots(\alpha+k-1)^n}{\left(k!\right)^n\theta^{kn}}\right] + k\sum_{i=1}^n \log s_i - (\alpha+k)\sum_{i=1}^n \log\left(1+\frac{s_i}{\theta}\right).$$

The partial derivatives of this with respect to α and θ are

$$\frac{\partial \log L\left(\alpha,\theta\right)}{\partial\alpha} = \frac{n}{\alpha}+\frac{n}{\alpha+1}+\cdots+\frac{n}{\alpha+k-1}-\sum_{i=1}^n\log\left(1+\frac{s_i}{\theta}\right)$$

and

$$\frac{\partial \log L(\alpha, \theta)}{\partial \theta} = -\frac{kn}{\theta} + \frac{\alpha + k}{\theta^2} \sum_{i=1}^{n} s_i \left(1 + \frac{s_i}{\theta}\right)^{-1}.$$

The maximum likelihood estimates of α and θ are the simultaneous solutions of

$$\frac{n}{\alpha} + \frac{n}{\alpha + 1} + \cdots + \frac{n}{\alpha + k - 1} = \sum_{i=1}^{n} \log\left(1 + \frac{s_i}{\theta}\right)$$

and

$$\frac{kn}{\theta} = \frac{\alpha + k}{\theta^2} \sum_{i=1}^{n} s_i \left(1 + \frac{s_i}{\theta}\right)^{-1}.$$

(4 marks)

UNSEEN