

**SOLUTIONS TO
MATH48181
EXTREME VALUES
AND FINANCIAL RISK EXAM**

Solutions to Question A1

a) The marginal cdfs of

$$F_{X,Y}(x, y) = 1 - \exp(-x^\alpha) - \exp(-y^\alpha) + [\exp(x^\alpha) + \exp(y^\alpha) - 1]^{-1}$$

are

$$F_X(x) = F_{X,Y}(x, \infty) = 1 - \exp(-x^\alpha)$$

and

$$F_Y(y) = F_{X,Y}(\infty, y) = 1 - \exp(-y^\alpha).$$

(3 marks)

UNSEEN

b) First note that $w(F_X) = +\infty$. F_X belongs to the Gumbel max domain of attraction since

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1 - F_X(t + x\gamma(t))}{1 - F_X(t)} &= \lim_{t \rightarrow \infty} \frac{e^{-(t+x\gamma(t))^\alpha}}{e^{-t^\alpha}} \\ &= \lim_{t \rightarrow \infty} e^{t^\alpha - (t+x\gamma(t))^\alpha} \\ &= \lim_{t \rightarrow \infty} e^{t^\alpha [1 - (1 + \frac{x\gamma(t)}{t})^\alpha]} \\ &= \lim_{t \rightarrow \infty} e^{t^\alpha [1 - (1 + \alpha \frac{x\gamma(t)}{t})]} \\ &= \lim_{t \rightarrow \infty} e^{-\alpha x t^{\alpha-1} \gamma(t)} \\ &= e^{-x} \end{aligned}$$

if $\gamma(t) = \alpha^{-1} t^{1-\alpha}$.

(3 marks)

c) First note that $w(F_Y) = +\infty$. F_Y belongs to the Gumbel max domain of attraction since

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1 - F_Y(t + y\gamma(t))}{1 - F_Y(t)} &= \lim_{t \rightarrow \infty} \frac{e^{-(t+y\gamma(t))^\alpha}}{e^{-t^\alpha}} \\ &= \lim_{t \rightarrow \infty} e^{t^\alpha - (t+y\gamma(t))^\alpha} \\ &= \lim_{t \rightarrow \infty} e^{t^\alpha [1 - (1 + \frac{y\gamma(t)}{t})^\alpha]} \\ &= \lim_{t \rightarrow \infty} e^{t^\alpha [1 - (1 + \alpha \frac{y\gamma(t)}{t})]} \\ &= \lim_{t \rightarrow \infty} e^{-\alpha y t^{\alpha-1} \gamma(t)} \\ &= e^{-y} \end{aligned}$$

if $\gamma(t) = \alpha^{-1}t^{1-\alpha}$.

(2 marks)

SEEN

d) Use the rule $a_n = \gamma^{-1}(F_X^{-1}(1 - n^{-1}))$ and $b_n = F_X^{-1}(1 - n^{-1})$. Inverting

$$F_X(x) = 1 - \exp(-x^\alpha) = 1 - n^{-1},$$

we obtain

$$F_X^{-1}(1 - n^{-1}) = (\log n)^{1/\alpha}.$$

So,

$$a_n = \frac{1}{\alpha}(\log n)^{\frac{1}{\alpha}-1}$$

we obtain

$$b_n = (\log n)^{1/\alpha}$$

(3 marks)

UNSEEN

e) Use the rule $c_n = \gamma^{-1}(F_Y^{-1}(1 - n^{-1}))$ and $d_n = F_Y^{-1}(1 - n^{-1})$. Inverting

$$F_Y(y) = 1 - \exp(-y^\alpha) = 1 - n^{-1},$$

we obtain

$$F_Y^{-1}(1 - n^{-1}) = (\log n)^{1/\alpha}.$$

So,

$$c_n = \frac{1}{\alpha}(\log n)^{\frac{1}{\alpha}-1}$$

we obtain

$$d_n = (\log n)^{1/\alpha}$$

(2 marks)

UNSEEN

g) The limiting cdf is

$$\begin{aligned}
& \lim_{n \rightarrow \infty} F_{X,Y}(a_n x + b_n, c_n y + d_n) \\
&= \lim_{n \rightarrow \infty} \left\{ 1 - \exp \left\{ - \left[\frac{1}{\alpha} (\log n)^{\frac{1}{\alpha}-1} x + (\log n)^{\frac{1}{\alpha}} \right]^\alpha \right\} \right. \\
&\quad \left. - \exp \left\{ - \left[\frac{1}{\alpha} (\log n)^{\frac{1}{\alpha}-1} y + (\log n)^{\frac{1}{\alpha}} \right]^\alpha \right\} \right. \\
&\quad \left. + \left[\exp \left\{ \left[\frac{1}{\alpha} (\log n)^{\frac{1}{\alpha}-1} x + (\log n)^{\frac{1}{\alpha}} \right]^\alpha \right\} + \exp \left\{ \left[\frac{1}{\alpha} (\log n)^{\frac{1}{\alpha}-1} y + (\log n)^{\frac{1}{\alpha}} \right]^\alpha \right\} - 1 \right]^{-1} \right\}^n \\
&= \lim_{n \rightarrow \infty} \left\{ 1 - \exp \left\{ - \log n \left[\frac{1}{\alpha} (\log n)^{-1} x + 1 \right]^\alpha \right\} \right. \\
&\quad \left. - \exp \left\{ - \log n \left[\frac{1}{\alpha} (\log n)^{-1} y + 1 \right]^\alpha \right\} \right. \\
&\quad \left. + \left[\exp \left\{ \log n \left[\frac{1}{\alpha} (\log n)^{-1} x + 1 \right]^\alpha \right\} + \exp \left\{ \log n \left[\frac{1}{\alpha} (\log n)^{-1} y + 1 \right]^\alpha \right\} - 1 \right]^{-1} \right\}^n \\
&= \lim_{n \rightarrow \infty} \left\{ 1 - n^{-\left[\frac{1}{\alpha} (\log n)^{-1} x + 1 \right]^\alpha} - n^{-\left[\frac{1}{\alpha} (\log n)^{-1} y + 1 \right]^\alpha} + \left[n^{\left[\frac{1}{\alpha} (\log n)^{-1} x + 1 \right]^\alpha} + n^{\left[\frac{1}{\alpha} (\log n)^{-1} y + 1 \right]^\alpha} - 1 \right]^{-1} \right\}^n \\
&= \lim_{n \rightarrow \infty} \left\{ 1 - \frac{\exp(-x)}{n} - \frac{\exp(-y)}{n} + [n \exp(x) + n \exp(y) - 1]^{-1} \right\}^n \\
&= \lim_{n \rightarrow \infty} \left\{ 1 - \frac{\exp(-x)}{n} - \frac{\exp(-y)}{n} + \frac{1}{n} \left[\exp(x) + \exp(y) - \frac{1}{n} \right]^{-1} \right\}^n \\
&= \lim_{n \rightarrow \infty} \exp \left\{ - \exp(-x) - \exp(-y) + \left[\exp(x) + \exp(y) - \frac{1}{n} \right]^{-1} \right\} \\
&= \exp \left\{ - \exp(-x) - \exp(-y) + [\exp(x) + \exp(y)]^{-1} \right\}.
\end{aligned}$$

(5 marks)

UNSEEN

h) No, the extremes are not completely independent since

$$\begin{aligned}
\lim_{n \rightarrow \infty} F_{X,Y}(a_n x + b_n, c_n y + d_n) &= \exp \left\{ - \exp(-x) - \exp(-y) + [\exp(x) + \exp(y)]^{-1} \right\} \\
&\neq \exp \left\{ - \exp(-x) - \exp(-y) \right\} \\
&= \lim_{n \rightarrow \infty} F_X(a_n x + b_n) \lim_{n \rightarrow \infty} F_Y(c_n y + d_n).
\end{aligned}$$

(2 marks)

UNSEEN

Solutions to Question A2

$C(u_1, u_2)$ is a valid copula if

$$C(u, 0) = 0,$$

$$C(0, u) = 0,$$

$$C(1, u) = u,$$

$$C(u, 1) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) \geq 0.$$

(4 marks)

SEEN

a) for the copula defined by $C(u_1, u_2) = \alpha C_1(u_1, u_2) + (1 - \alpha)C_2(u_1, u_2)$, we have

$$C(u, 0) = \alpha C_1(u, 0) + (1 - \alpha)C_2(u, 0) = \alpha \cdot 0 + (1 - \alpha) \cdot 0 = 0,$$

$$C(0, u) = \alpha C_1(0, u) + (1 - \alpha)C_2(0, u) = \alpha \cdot 0 + (1 - \alpha) \cdot 0 = 0,$$

$$C(1, u) = \alpha C_1(1, u) + (1 - \alpha)C_2(1, u) = \alpha \cdot 1 + (1 - \alpha) \cdot 1 = 1,$$

$$C(u, 1) = \alpha C_1(u, 1) + (1 - \alpha)C_2(u, 1) = \alpha \cdot 1 + (1 - \alpha) \cdot 1 = 1,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \alpha \frac{\partial}{\partial u_1} C_1(u_1, u_2) + (1 - \alpha) \frac{\partial}{\partial u_1} C_2(u_1, u_2) \geq 0 + 0 = 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \alpha \frac{\partial}{\partial u_2} C_1(u_1, u_2) + (1 - \alpha) \frac{\partial}{\partial u_2} C_2(u_1, u_2) \geq 0 + 0 = 0.$$

(4 marks)

UNSEEN

b) for the copula defined by $C(u_1, u_2) = \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2}$, we have

$$C(u, 0) = \frac{u \cdot 0}{u + 0 - u \cdot 0} = 0,$$

$$C(0, u) = \frac{0 \cdot u}{0 + u - 0 \cdot u} = 0,$$

$$C(1, u) = \frac{1 \cdot u}{1 + u - 1 \cdot u} = u,$$

$$C(u, 1) = \frac{u \cdot 1}{u + 1 - u \cdot 1} = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \frac{\partial}{\partial u_1} \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2} = \frac{(u_1 + u_2 - u_1 u_2) u_1 - u_1 u_2 (1 - u_1)}{(u_1 + u_2 - u_1 u_2)^2} = \frac{u_1^2}{(u_1 + u_2 - u_1 u_2)^2} \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \frac{\partial}{\partial u_2} \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2} = \frac{(u_1 + u_2 - u_1 u_2) u_2 - u_1 u_2 (1 - u_2)}{(u_1 + u_2 - u_1 u_2)^2} = \frac{u_2^2}{(u_1 + u_2 - u_1 u_2)^2} \geq 0.$$

(4 marks)

UNSEEN

c) for the copula defined by $C(u_1, u_2) = u_1 u_2 + \alpha u_1 u_2 (1 - u_1)(1 - u_2)$, we have

$$C(u, 0) = u \cdot 0 + \alpha u \cdot 0 \cdot (1 - u)(1 - 0) = 0,$$

$$C(0, u) = 0 \cdot u + \alpha 0 \cdot u \cdot (1 - 0)(1 - u) = 0,$$

$$C(1, u) = 1 \cdot u + \alpha 1 \cdot u \cdot (1 - 1)(1 - u) = u,$$

$$C(u, 1) = u \cdot 1 + \alpha u \cdot 1 \cdot (1 - u)(1 - 1) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = u_2 + \alpha u_2 (1 - 2u_1)(1 - u_2) = u_2 [1 + \alpha (1 - 2u_1)(1 - u_2)] \geq 0$$

since $-1 \leq \alpha(1 - 2u_1)(1 - u_2) \leq 1$ and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = u_1 + \alpha u_1(1 - 2u_2)(1 - u_1) = u_1 [1 + \alpha(1 - 2u_2)(1 - u_1)] \geq 0$$

since $-1 \leq \alpha(1 - 2u_2)(1 - u_1) \leq 1$.

(4 marks)

UNSEEN

d) for the copula defined by $C(u_1, u_2) = \begin{cases} \max(u_1 + u_2 - 1, t), & \text{if } t \leq u_1 \leq 1, t \leq u_2 \leq 1, \\ \min(u_1, u_2), & \text{otherwise,} \end{cases}$
we have

$$C(u, 0) = \min(u, 0) = 0,$$

$$C(0, u) = \min(0, u) = 0,$$

$$C(1, u) = \begin{cases} \max(1 + u - 1, t), & \text{if } t \leq u \leq 1, \\ \min(1, u), & \text{otherwise} \end{cases} = u,$$

$$C(u, 1) = \begin{cases} \max(u + 1 - 1, t), & \text{if } t \leq u \leq 1, \\ \min(u, 1), & \text{otherwise} \end{cases} = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \begin{cases} \frac{\partial}{\partial u_1} \max(u_1 + u_2 - 1, t), & \text{if } t \leq u_1 \leq 1, t \leq u_2 \leq 1, \\ \frac{\partial}{\partial u_1} \min(u_1, u_2), & \text{otherwise} \end{cases} \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \begin{cases} \frac{\partial}{\partial u_2} \max(u_1 + u_2 - 1, t), & \text{if } t \leq u_1 \leq 1, t \leq u_2 \leq 1, \\ \frac{\partial}{\partial u_2} \min(u_1, u_2), & \text{otherwise} \end{cases} \geq 0.$$

(4 marks)

UNSEEN

Solutions to Question A3

a) We can write

$$\bar{F}(x, y) = \exp \left\{ -(x + y) \sum_{i=1}^k \alpha_i A_i \left(\frac{y}{x + y} \right) \right\}$$

for $x > 0$, $y > 0$ and $\alpha_i \geq 0$ sum to one. This is in the form of

$$\bar{F}(x, y) = \exp \left[-(x + y) A \left(\frac{y}{x + y} \right) \right]$$

with

$$A(w) = \sum_{i=1}^k \alpha_i A_i(w).$$

We now check the conditions for $A(\cdot)$. Clearly,

$$A(0) = \sum_{i=1}^k \alpha_i A_i(0) = \sum_{i=1}^k \alpha_i \cdot 1 = 1$$

and

$$A(1) = \sum_{i=1}^k \alpha_i A_i(1) = \sum_{i=1}^k \alpha_i \cdot 1 = 1.$$

Also $A(w) \geq 0$ since $A_i(w) \geq 0$ for all w and every k .

Also since each $A_i(w) \leq 1$,

$$A(w) = \sum_{i=1}^k \alpha_i A_i(w) \leq \sum_{i=1}^k \alpha_i \cdot 1 = 1.$$

Also since each $A_i(w) \geq \max(w, 1 - w)$,

$$A(w) = \sum_{i=1}^k \alpha_i A_i(w) \geq \sum_{i=1}^k \alpha_i \cdot \max(w, 1 - w) = \max(w, 1 - w).$$

Also since each $A_i(w)$ is convex,

$$A''(w) = \sum_{i=1}^k \alpha_i A_i''(w) \geq 0.$$

(7 marks)

UNSEEN

b) Note that

$$\bar{F}(x, 0) = \exp \left\{ -(x + 0) \sum_{i=1}^k \alpha_i A_i(0) \right\} = \exp(-x)$$

and

$$\bar{F}(0, y) = \exp \left\{ -(0 + y) \sum_{i=1}^k \alpha_i A_i(0) \right\} = \exp(-y).$$

So, the joint cdf is

$$F(x, y) = 1 - \exp(-x) - \exp(-y) + \exp \left\{ -(x + y) \sum_{i=1}^k \alpha_i A_i \left(\frac{y}{x + y} \right) \right\}.$$

(1 marks)

UNSEEN

c) the derivative of joint cdf with respect to x is

$$\frac{\partial F(x, y)}{\partial x} = \exp(-x) + \bar{F}(x, y) \left\{ - \sum_{i=1}^k \alpha_i A_i \left(\frac{y}{x + y} \right) + \frac{y}{x + y} \sum_{i=1}^k \alpha_i A_i' \left(\frac{y}{x + y} \right) \right\},$$

so the conditional cdf of Y given $X = x$ is

$$F(y | x) = 1 + \exp(x) \bar{F}(x, y) \left\{ - \sum_{i=1}^k \alpha_i A_i \left(\frac{y}{x + y} \right) + \frac{y}{x + y} \sum_{i=1}^k \alpha_i A_i' \left(\frac{y}{x + y} \right) \right\}.$$

(4 marks)

UNSEEN

d) the derivative of joint cdf with respect to y is

$$\frac{\partial F(x, y)}{\partial y} = \exp(-y) + \bar{F}(x, y) \left\{ - \sum_{i=1}^k \alpha_i A_i \left(\frac{y}{x + y} \right) - \frac{x}{x + y} \sum_{i=1}^k \alpha_i A_i' \left(\frac{y}{x + y} \right) \right\},$$

so the conditional cdf of Y given $X = x$ is

$$F(x | y) = 1 + \exp(y)\bar{F}(x, y) \left\{ - \sum_{i=1}^k \alpha_i A_i \left(\frac{y}{x+y} \right) - \frac{x}{x+y} \sum_{i=1}^k \alpha_i A_i' \left(\frac{y}{x+y} \right) \right\}.$$

(4 marks)

UNSEEN

e) the derivative of joint cdf with respect to x and y is

$$\begin{aligned} f(x, y) &= \bar{F}(x, y) \left\{ - \sum_{i=1}^k \alpha_i A_i \left(\frac{y}{x+y} \right) + \frac{y}{x+y} \sum_{i=1}^k \alpha_i A_i' \left(\frac{y}{x+y} \right) \right\} \\ &\cdot \left\{ - \sum_{i=1}^k \alpha_i A_i \left(\frac{y}{x+y} \right) - \frac{x}{x+y} \sum_{i=1}^k \alpha_i A_i' \left(\frac{y}{x+y} \right) \right\} \\ &+ \bar{F}(x, y) \frac{xy}{(x+y)^2} \sum_{i=1}^k \alpha_i A_i'' \left(\frac{y}{x+y} \right). \end{aligned}$$

(4 marks)

UNSEEN

Solutions to Question B1

a) The cumulative distribution function of T conditional on $N = n$ is

$$\begin{aligned}\Pr(T \leq t \mid N = n) &= \Pr(\max(X_1, \dots, X_N) \leq t \mid N = n) \\ &= \Pr(\max(X_1, \dots, X_n) \leq t \mid N = n) \\ &= \Pr(X_1 \leq t, \dots, X_n \leq t) \\ &= \Pr(X_1 \leq t) \cdots \Pr(X_n \leq t) \\ &= \frac{t+a}{2a} \cdots \frac{t+a}{2a} \\ &= \left(\frac{t+a}{2a}\right)^n.\end{aligned}$$

(4 marks)

UNSEEN

b) The unconditional cumulative distribution function of T is

$$\begin{aligned}\Pr(T \leq t) &= \sum_{n=0}^{\infty} \Pr(T \leq t \mid N = n) \frac{\theta^n \exp(-\theta)}{n!} \\ &= \sum_{n=0}^{\infty} \left(\frac{t+a}{2a}\right)^n \frac{\theta^n \exp(-\theta)}{n!} \\ &= \exp(-\theta) \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\theta(t+a)}{2a}\right)^n \\ &= \exp\left(-\theta + \frac{\theta(t+a)}{2a}\right) \\ &= \exp\left(\frac{\theta(t-a)}{2a}\right).\end{aligned}$$

(4 marks)

UNSEEN

c) The unconditional probability density function of T is

$$f_T(t) = \frac{\theta}{2a} \exp\left(\frac{\theta(t-a)}{2a}\right).$$

(1 marks)

UNSEEN

d) The moment generating function of T is

$$\begin{aligned}
 M_T(s) &= \frac{\theta}{2a} \exp\left(-\frac{\theta}{2}\right) \int_{-a}^a \exp\left(st + \frac{\theta t}{2a}\right) dt \\
 &= \frac{\theta}{2a} \exp\left(-\frac{\theta}{2}\right) \left(s + \frac{\theta}{2a}\right)^{-1} \left[\exp\left(st + \frac{\theta t}{2a}\right)\right]_{-a}^a \\
 &= \frac{\theta}{2a} \exp\left(-\frac{\theta}{2}\right) \left(s + \frac{\theta}{2a}\right)^{-1} \left[\exp\left(sa + \frac{\theta}{2}\right) - \exp\left(-sa - \frac{\theta}{2}\right)\right].
 \end{aligned}$$

(4 marks)

UNSEEN

e) Setting

$$\exp\left(\frac{\theta(t-a)}{2a}\right) \cdot = p$$

and solving for t , we obtain value at risk of T as

$$\text{VaR}_p(T) = a + \frac{2a}{\theta} \log p.$$

(3 marks)

UNSEEN

f) The expected shortfall T is

$$\begin{aligned}
 \text{ES}_p(T) &= a + \frac{2a}{\theta p} \int_0^p \log x dx \\
 &= a + \frac{2a}{\theta p} \{[t \log t]_0^p - p\} \\
 &= a + \frac{2a}{\theta p} \{p \log p - p\} \\
 &= a + \frac{2a}{\theta} \{\log p - 1\}.
 \end{aligned}$$

(4 marks)

UNSEEN

Solutions to Question B2

If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cdf of a normalized version of M_n converges to G , i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n(a_n x + b_n) \rightarrow G(x) \quad (1)$$

as $n \rightarrow \infty$ then G must be of the same type as (cdf's G and G^* are of the same type if $G^*(x) = G(ax + b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$\begin{aligned} I & : \Lambda(x) = \exp\{-\exp(-x)\}, \quad x \in \mathfrak{R}; \\ II & : \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\{-x^{-\alpha}\} & \text{if } x \geq 0 \end{cases} \\ & \quad \text{for some } \alpha > 0; \\ III & : \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^\alpha\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \\ & \quad \text{for some } \alpha > 0. \end{aligned}$$

(4 marks)

SEEN

The necessary and sufficient conditions for the three extreme value distributions are:

$$\begin{aligned} I & : \exists \gamma(t) > 0 \text{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = \exp(-x), \quad x \in \mathfrak{R}, \\ II & : w(F) = \infty \text{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \quad x > 0, \\ III & : w(F) < \infty \text{ and } \lim_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} = x^\alpha, \quad x > 0. \end{aligned}$$

(4 marks)

SEEN

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function say $h(t)$ such that

$$\lim_{t \rightarrow w(G)} \frac{1 - G(t + xh(t))}{1 - G(t)} = e^{-x}$$

for every $x \in (-\infty, \infty)$. But

$$\begin{aligned}
\lim_{t \rightarrow w(F)} \frac{1 - F(t + xh(t))}{1 - F(t)} &= \lim_{t \rightarrow w(F)} \frac{(1 + xh'(t)) f(t + xh(t))}{f(t)} \\
&= \lim_{t \rightarrow w(F)} \frac{(1 + xh'(t)) g(t + xh(t)) G^{a-1}(t + xh(t)) [1 - G(t + xh(t))]^{b-1}}{g(t) G^{a-1}(t) [1 - G(t)]^{b-1}} \\
&= \lim_{t \rightarrow w(F)} \frac{(1 + xh'(t)) g(t + xh(t)) [1 - G(t + xh(t))]^{b-1}}{g(t) [1 - G(t)]^{b-1}} \\
&= \lim_{t \rightarrow w(F)} \frac{(1 + xh'(t)) g(t + xh(t)) [1 - G(t + xh(t))]^{b-1}}{g(t) [1 - G(t)]^{b-1}} \\
&= \lim_{t \rightarrow w(F)} \frac{1 - G(t + xh(t)) [1 - G(t + xh(t))]^{b-1}}{1 - G(t) [1 - G(t)]^{b-1}} \\
&= \lim_{t \rightarrow w(F)} \left[\frac{1 - G(t + xh(t))}{1 - G(t)} \right]^b \\
&= \exp(-bx)
\end{aligned}$$

for every $x \in (-\infty, \infty)$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n \rightarrow \infty} P \left(\frac{M_n - b_n}{a_n} \leq x \right) = \exp[-\exp(-bx)]$$

for some suitable norming constants $a_n > 0$ and b_n .

(4 marks)

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t \rightarrow \infty} \frac{1 - G(tx)}{1 - G(t)} = x^{-\beta}$$

for every $x > 0$. But

$$\begin{aligned}
\lim_{t \rightarrow w(F)} \frac{1 - F(tx)}{1 - F(t)} &= \lim_{t \rightarrow w(F)} \frac{xf(tx)}{f(t)} \\
&= \lim_{t \rightarrow w(F)} \frac{xg(tx) G^{a-1}(tx) [1 - G(tx)]^{b-1}}{g(t) G^{a-1}(t) [1 - G(t)]^{b-1}} \\
&= \lim_{t \rightarrow w(F)} \frac{xg(tx) [1 - G(tx)]^{b-1}}{g(t) [1 - G(t)]^{b-1}} \\
&= \lim_{t \rightarrow w(F)} \frac{xg(tx) [1 - G(tx)]^{b-1}}{g(t) [1 - G(t)]^{b-1}} \\
&= \lim_{t \rightarrow w(F)} \frac{1 - G(tx) [1 - G(tx)]^{b-1}}{1 - G(t) [1 - G(t)]^{b-1}} \\
&= \lim_{t \rightarrow w(F)} \left[\frac{1 - G(t + xh(t))}{1 - G(t)} \right]^b \\
&= x^{-b\beta}
\end{aligned}$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n \rightarrow \infty} P \left(\frac{M_n - b_n}{a_n} \leq x \right) = \exp(-x^{-b\beta})$$

for some suitable norming constants $a_n > 0$ and b_n .

(4 marks)

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t \rightarrow 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} = x^\beta$$

for every $x > 0$. But

$$\begin{aligned}
\lim_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} &= \lim_{t \downarrow 0} \frac{xf(w(F) - tx)}{f(w(F) - t)} \\
&= \lim_{t \downarrow 0} \frac{xg(w(F) - tx) G^{a-1}(w(F) - tx) [1 - G(w(F) - tx)]^{b-1}}{g(t) G^{a-1}(w(F) - t) [1 - G(w(F) - t)]^{b-1}} \\
&= \lim_{t \downarrow 0} \frac{xg(w(F) - tx) [1 - G(w(F) - tx)]^{b-1}}{g(w(F) - t) [1 - G(w(F) - t)]^{b-1}} \\
&= \lim_{t \downarrow 0} \frac{xg(w(F) - tx) [1 - G(w(F) - tx)]^{b-1}}{g(w(F) - t) [1 - G(w(F) - t)]^{b-1}} \\
&= \lim_{t \downarrow 0} \frac{1 - G(w(F) - tx) [1 - G(w(F) - tx)]^{b-1}}{1 - G(w(F) - t) [1 - G(w(F) - t)]^{b-1}} \\
&= \lim_{t \downarrow 0} \left[\frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \right]^b \\
&= x^{b\beta}
\end{aligned}$$

for every $x < 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{n \rightarrow \infty} P \left(\frac{M_n - b_n}{a_n} \leq x \right) = \exp(-(-x)^{b\beta})$$

for some suitable norming constants $a_n > 0$ and b_n .

(4 marks)

UNSEEN

Solutions to Question B3

a) Note that $w(F) = N$ and

$$\begin{aligned}\frac{\Pr(X = w(F))}{1 - F(w(F) - 1)} &= \frac{\Pr(X = N)}{1 - F(N - 1)} \\ &= \frac{\Pr(X = N)}{1 - \Pr(X = 1) - \Pr(X = 2) - \dots - \Pr(X = N - 1)} \\ &= \frac{\frac{1}{N}}{1 - \frac{1}{N} - \frac{1}{N} - \dots - \frac{1}{N}} \\ &= \frac{\frac{1}{N}}{1 - \frac{N-1}{N}} \\ &= 1.\end{aligned}$$

Hence, there can be no non-degenerate limit.

(4 marks)

UNSEEN

b) Note that $w(F) = n$ and

$$\begin{aligned}\frac{\Pr(X = w(F))}{1 - F(w(F) - 1)} &= \frac{\Pr(X = n)}{1 - F(n - 1)} \\ &= \frac{\Pr(X = n)}{1 - \Pr(X \leq n - 1)} \\ &= \frac{\Pr(X = n)}{\Pr(X = n)} \\ &= 1.\end{aligned}$$

Hence, there can be no non-degenerate limit.

c) Note that $w(F) = b$. The corresponding cdf is

$$F(x) = \frac{\log x - \log a}{\log b - \log a}.$$

Note that

$$\begin{aligned}
\lim_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} &= \lim_{t \downarrow 0} \frac{1 - F(b - tx)}{1 - F(b - t)} \\
&= \lim_{t \downarrow 0} \frac{1 - \frac{\log(b-tx) - \log a}{\log b - \log a}}{1 - \frac{\log(b-t) - \log a}{\log b - \log a}} \\
&= \lim_{t \downarrow 0} \frac{\log b - \log(b - tx)}{\log b - \log(b - t)} \\
&= \lim_{t \downarrow 0} \frac{\frac{x}{b-tx}}{\frac{1}{b-t}} \\
&= x.
\end{aligned}$$

So, $F(x)$ belongs to the Weibull domain of attraction.

(4 marks)

UNSEEN

d) Note that $w(F) = \infty$. Then

$$\begin{aligned}
\lim_{t \rightarrow \infty} \frac{1 - F(t + xg(t))}{1 - F(t)} &= \lim_{t \rightarrow \infty} \frac{1 - \{1 - \exp[-(t + xg(t))^2]\}^a}{1 - \{1 - \exp[-t^2]\}^a} \\
&= \lim_{t \rightarrow \infty} \frac{1 - \{1 - a \exp[-(t + xg(t))^2]\}}{1 - \{1 - a \exp[-t^2]\}} \\
&= \lim_{t \rightarrow \infty} \frac{\exp[-(t + xg(t))^2]}{\exp[-t^2]} \\
&= \lim_{t \rightarrow \infty} \exp[-2txg(t) - x^2g^2(t)] \\
&= \exp(-x)
\end{aligned}$$

if $g(t) = 1/(2t)$. So, $F(x) = [1 - \exp(-x^2)]^a$ belongs to the Gumbel domain of attraction.

(4 marks)

UNSEEN

e) Note that $w(F) = \infty$. Then

$$\begin{aligned}\lim_{t \rightarrow \infty} \frac{1 - F(tx)}{1 - F(t)} &= \lim_{t \uparrow \infty} \frac{1 - 1 + [1 - \exp(-(tx)^{-1})]^a}{1 - 1 + [1 - \exp(-t^{-1})]^a} \\ &= \lim_{t \uparrow \infty} \frac{[1 - \exp(-(tx)^{-1})]^a}{[1 - \exp(-t^{-1})]^a} \\ &= \lim_{t \uparrow \infty} \left[\frac{1 - \exp(-(tx)^{-1})}{1 - \exp(-t^{-1})} \right]^a \\ &= \lim_{t \uparrow \infty} \left[\frac{1 - 1 + (tx)^{-1}}{1 - 1 + t^{-1}} \right]^a \\ &= x^{-a}.\end{aligned}$$

. So, the cdf $F(x) = 1 - [1 - \exp(-x^{-1})]^a$ belongs to the Fréchet domain of attraction.

(4 marks)

UNSEEN

Solutions to Question B4

(a) If X is an absolutely continuous random variable with cdf $F(\cdot)$ then

$$\text{VaR}_p(X) = F^{-1}(p)$$

and

$$\text{ES}_p(X) = \frac{1}{p} \int_0^p F^{-1}(v) dv.$$

(2 marks)

SEEN

(b) (i) The corresponding cdf is

$$F(x) = \int_{-a}^x \frac{3y^2}{2a^3} dy = \frac{3}{2a^3} \left[\frac{y^3}{3} \right]_{-a}^x = \frac{3}{2a^3} \left[\frac{x^3 + a^3}{3} \right] = \frac{x^3 + a^3}{2a^3}$$

for $-a < x < a$;

(2 marks)

UNSEEN

(b) (ii) Inverting

$$F(x) = \frac{x^3 + a^3}{2a^3} = p,$$

we obtain $\text{VaR}_p(X) = (2p - 1)^{1/3} a$.

(1 marks)

UNSEEN

(b) (iii) The expected shortfall is

$$\text{ES}_p(X) = \frac{1}{p} \int_0^p (2v - 1)^{1/3} dv = \frac{1}{p} \left[\frac{3(2v - 1)^{4/3}}{8} \right]_0^p = \frac{3a}{8p} [(2p - 1)^{4/3} - 1].$$

(2 marks)

UNSEEN

c) (i) The likelihood function of a is

$$\begin{aligned}
 L(a) &= \frac{3^n}{2^n a^{3n}} \prod_{i=1}^n [X_i^2 I\{-a < X_i < a\}] \\
 &= \frac{3^n}{2^n a^{3n}} \left(\prod_{i=1}^n X_i \right)^2 \prod_{i=1}^n I\{-a < X_i < a\} \\
 &= \frac{3^n}{2^n a^{3n}} \left(\prod_{i=1}^n X_i \right)^2 I\{\max(X_1, \dots, X_n) < a, \min(X_1, \dots, X_n) > -a\} \\
 &= \frac{3^n}{2^n a^{3n}} \left(\prod_{i=1}^n X_i \right)^2 I\{a > \max(X_1, \dots, X_n), a > -\min(X_1, \dots, X_n)\} \\
 &= \frac{3^n}{2^n a^{3n}} \left(\prod_{i=1}^n X_i \right)^2 I\{a > \max[\max(X_1, \dots, X_n), -\min(X_1, \dots, X_n)]\}.
 \end{aligned}$$

(3 marks)

UNSEEN

c) (ii) Note that $L(a)$ is a decreasing function over a and $a > \max[\max(X_1, \dots, X_n), -\min(X_1, \dots, X_n)]$. Hence, the mle of a is $\max[\max(X_1, \dots, X_n), -\min(X_1, \dots, X_n)]$.

(1 marks)

UNSEEN

c) (iii) The mles of VaR and ES are $\widehat{\text{VaR}}_p(X) = (2p - 1)^{1/3} \widehat{a}$ and

$$\widehat{\text{ES}}_p(X) = \frac{3\widehat{a}}{8p} [(2p - 1)^{4/3} - 1],$$

where $\widehat{a} = \max[\max(X_1, \dots, X_n), -\min(X_1, \dots, X_n)]$.

(2 marks)

UNSEEN

c (iv) Let $Z = \max[\max(X_1, \dots, X_n), -\min(X_1, \dots, X_n)]$. The cdf of Z is

$$\begin{aligned}
 F_Z(z) &= \Pr(\max[\max(X_1, \dots, X_n), -\min(X_1, \dots, X_n)] \leq z) \\
 &= \Pr(\max(X_1, \dots, X_n) \leq z, -\min(X_1, \dots, X_n) \leq z) \\
 &= \Pr(\max(X_1, \dots, X_n) \leq z, \min(X_1, \dots, X_n) \geq -z) \\
 &= \Pr(X_1 \leq z, \dots, X_n \leq z, X_1 \geq -z, \dots, X_n \geq -z) \\
 &= \Pr(-z \leq X_1 \leq z, \dots, -z \leq X_n \leq z) \\
 &= \Pr^n(-z \leq X \leq z) \\
 &= [\Pr(X \leq z) - \Pr(X \leq -z)]^n \\
 &= \left[\frac{z^3}{a^3}\right]^n
 \end{aligned}$$

for $z > 0$. The corresponding pdf is

$$f_Z(z) = 3na^{-3n}z^{3n-1}$$

for $z > 0$.

(3 marks)

UNSEEN

c (v) The expected value of Z is

$$E(Z) = 3na^{-3n} \int_0^a z^{3n} dz = 3na^{-3n} \left[\frac{z^{3n+1}}{3n+1} \right]_0^a dz = 3na^{-3n} \frac{a^{3n+1}}{3n+1} = \frac{3na}{3n+1}.$$

Hence, \hat{a} is biased for a .

(2 marks)

UNSEEN

c (vi) Since

$$\text{Bias} \left[\widehat{\text{VaR}}_p(X) \right] = (2p-1)^{1/3} \text{Bias} [\hat{a}],$$

we see that $\widehat{\text{VaR}}_p(X)$ is biased for $\text{VaR}_p(X)$.

(1 marks)

UNSEEN

c (vii) Since

$$\text{Bias} \left[\widehat{\text{ES}}_p(X) \right] = \frac{3}{8p} \left[(2p-1)^{4/3} - 1 \right] \text{Bias} [\hat{a}],$$

we see that $\widehat{ES}_p(X)$ is biased for $ES_p(X)$.

(1 marks)

UNSEEN

Solutions to Question B5

a) Note that

$$\frac{\partial \bar{F}}{\partial x_1} = -\frac{\alpha}{\theta} \left(1 + \frac{1}{\theta} \sum_{i=1}^k x_i \right)^{-\alpha-1},$$

$$\frac{\partial^2 \bar{F}}{\partial x_1 \partial x_2} = \frac{\alpha(\alpha+1)}{\theta^2} \left(1 + \frac{1}{\theta} \sum_{i=1}^k x_i \right)^{-\alpha-2},$$

$$\frac{\partial^3 \bar{F}}{\partial x_1 \partial x_2 \partial x_3} = -\frac{\alpha(\alpha+1)(\alpha+3)}{\theta^3} \left(1 + \frac{1}{\theta} \sum_{i=1}^k x_i \right)^{-\alpha-3},$$

and so on. In general,

$$f(x_1, x_2, \dots, x_k) = \frac{\alpha(\alpha+1) \cdots (\alpha+k-1)}{\theta^k} \left(1 + \frac{1}{\theta} \sum_{i=1}^k x_i \right)^{-\alpha-k},$$

as required.

(4 marks)

UNSEEN

b) The pdf of S is

$$\begin{aligned} f_S(s) &= \int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{k-1}} f(x_1, x_2, \dots, t-x_1-x_2-\cdots-x_{k-1}) dx_k \cdots dx_2 dx_1 \\ &= \frac{\alpha(\alpha+1) \cdots (\alpha+k-1)}{\theta^k} \int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{k-1}} \left(1 + \frac{1}{\theta} \sum_{i=1}^k x_i \right)^{-\alpha-k} dx_k \cdots dx_2 dx_1 \\ &= \frac{\alpha(\alpha+1) \cdots (\alpha+k-1)}{\theta^k} \int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{k-1}} \left(1 + \frac{s}{\theta} \right)^{-\alpha-k} dx_k \cdots dx_2 dx_1 \\ &= \frac{\alpha(\alpha+1) \cdots (\alpha+k-1)}{\theta^k} \left(1 + \frac{s}{\theta} \right)^{-\alpha-k} \int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{k-1}} dx_k \cdots dx_2 dx_1 \\ &= \frac{\alpha(\alpha+1) \cdots (\alpha+k-1)}{k! \theta^k} s^k \left(1 + \frac{s}{\theta} \right)^{-\alpha-k}, \end{aligned}$$

as required.

(4 marks)

UNSEEN

c) The cdf of S is

$$\begin{aligned} F_S(s) &= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!\theta^k} \int_0^s x^k \left(1 + \frac{x}{\theta}\right)^{-\alpha-k} dx \\ &= \frac{\theta\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!} \int_1^{\theta/(s+\theta)} y^{\alpha-2}(1-y)^k dy \\ &= \frac{\theta\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!} \int_0^{s/(s+\theta)} (1-y)^{\alpha-2} y^k dy \\ &= \frac{\theta\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!} B_{s/(s+\theta)}(k+1, \alpha-1), \end{aligned}$$

as required.

(4 marks)

UNSEEN

d) The n th moment of S is

$$\begin{aligned} E(S^n) &= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!\theta^k} \int_0^\infty x^{n+k} \left(1 + \frac{x}{\theta}\right)^{-\alpha-k} dx \\ &= \frac{\theta^{n+1}\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!} \int_0^1 (1-y)^{n+k} y^{\alpha-n-2} dy \\ &= \frac{\theta^{n+1}\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!} B(\alpha-n-1, n+k+1). \end{aligned}$$

(4 marks)

UNSEEN

e) The joint likelihood function is

$$L(\alpha, \theta) = \frac{\alpha^n(\alpha+1)^n \cdots (\alpha+k-1)^n}{(k!)^n \theta^{kn}} \left(\prod_{i=1}^n s_i \right)^k \left[\prod_{i=1}^n \left(1 + \frac{s_i}{\theta}\right) \right]^{-\alpha-k}.$$

Its log is The joint likelihood function is

$$\log L(\alpha, \theta) = \log \left[\frac{\alpha^n(\alpha+1)^n \cdots (\alpha+k-1)^n}{(k!)^n \theta^{kn}} \right] + k \sum_{i=1}^n \log s_i - (\alpha+k) \sum_{i=1}^n \log \left(1 + \frac{s_i}{\theta}\right).$$

The partial derivatives of this with respect to α and θ are

$$\frac{\partial \log L(\alpha, \theta)}{\partial \alpha} = \frac{n}{\alpha} + \frac{n}{\alpha+1} + \cdots + \frac{n}{\alpha+k-1} - \sum_{i=1}^n \log \left(1 + \frac{s_i}{\theta}\right)$$

and

$$\frac{\partial \log L(\alpha, \theta)}{\partial \theta} = -\frac{kn}{\theta} + \frac{\alpha + k}{\theta^2} \sum_{i=1}^n s_i \left(1 + \frac{s_i}{\theta}\right)^{-1}.$$

The maximum likelihood estimates of α and θ are the simultaneous solutions of

$$\frac{n}{\alpha} + \frac{n}{\alpha + 1} + \cdots + \frac{n}{\alpha + k - 1} = \sum_{i=1}^n \log \left(1 + \frac{s_i}{\theta}\right)$$

and

$$\frac{kn}{\theta} = \frac{\alpha + k}{\theta^2} \sum_{i=1}^n s_i \left(1 + \frac{s_i}{\theta}\right)^{-1}.$$

(4 marks)

UNSEEN