SOLUTIONS TO
MATH38181
EXTREME VALUES
AND FINANCIAL RISK EXAM

Solutions to Question 1

a) The cumulative distribution function of T conditional on $N = n$ is

$$
\begin{aligned}
\Pr\left(T \le t \mid N = n\right) &= \Pr\left(\max\left(X_1, \ldots, X_N\right) \le t \mid N = n\right) \\
&= \Pr\left(\max\left(X_1, \ldots, X_n\right) \le t \mid N = n\right) \\
&= \Pr\left(X_1 \le t, \ldots, X_n \le t\right) \\
&= \Pr\left(X_1 \le t\right) \cdots \Pr\left(X_n \le t\right) \\
&= \frac{t+a}{2a} \cdots \frac{t+a}{2a} \\
&= \left(\frac{t+a}{2a}\right)^n.
\end{aligned}
$$

(4 marks)

UNSEEN

b) The unconditional cumulative distribution function of T is

$$
\begin{aligned}
\Pr(T \le t) &= \sum_{n=0}^{\infty} \Pr\left(T \le t \mid N = n\right) \frac{\theta^n \exp(-\theta)}{n!} \\
&= \sum_{n=0}^{\infty} \left(\frac{t+a}{2a}\right)^n \frac{\theta^n \exp(-\theta)}{n!} \\
&= \exp(-\theta) \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\theta(t+a)}{2a}\right)^n \\
&= \exp\left(-\theta + \frac{\theta(t+a)}{2a}\right) \\
&= \exp\left(\frac{\theta(t-a)}{2a}\right).
\end{aligned}
$$

(4 marks)

UNSEEN

c) The unconditional probability density function of T is

$$
f_T(t) = \frac{\theta}{2a} \exp\left(\frac{\theta(t-a)}{2a}\right).
$$

(1 marks)

UNSEEN

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d) The moment generating function of T is

$$
\begin{aligned}
M_T(s) &= \frac{\theta}{2a} \exp\left(-\frac{\theta}{2}\right) \int_{-a}^{a} \exp\left(st + \frac{\theta t}{2a}\right) dt \\
&= \frac{\theta}{2a} \exp\left(-\frac{\theta}{2}\right) \left(s + \frac{\theta}{2a}\right)^{-1} \left[\exp\left(st + \frac{\theta t}{2a}\right)\right]_{-a}^{a} \\
&= \frac{\theta}{2a} \exp\left(-\frac{\theta}{2}\right) \left(s + \frac{\theta}{2a}\right)^{-1} \left[\exp\left(sa + \frac{\theta}{2}\right) - \exp\left(-sa - \frac{\theta}{2}\right)\right].
\end{aligned}
$$

(4 marks)

UNSEEN

e) Setting

$$
\exp\left(\frac{\theta(t - a)}{2a}\right) . = p
$$

and solving for t, we obtain value at risk of T as

$$
\mathrm{VaR}_p(T) = a + \frac{2a}{\theta} \log p.
$$

(3 marks)

UNSEEN

f) The expected shortfall T is

$$
\begin{aligned}
\mathrm{ES}_p(T) &= a + \frac{2a}{\theta p} \int_0^p \log x\, dx \\
&= a + \frac{2a}{\theta p} \{[t \log t]_0^p - p\} \\
&= a + \frac{2a}{\theta p} \{p \log p - p\} \\
&= a + \frac{2a}{\theta} \{\log p - 1\}.
\end{aligned}
$$

(4 marks)

UNSEEN

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Solutions to Question 2

If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cdf of a normalized version of M_n converges to G, i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n\left(a_n x + b_n\right) \rightarrow G(x) \tag{1}$$

as $n \rightarrow \infty$ then G must be of the same type as (cdfs G and G^* are of the same type if $G^*(x) = G(ax + b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$I \quad : \quad \Lambda(x) = \exp\left\{-\exp(-x)\right\}, \qquad x \in \Re;$$

$$II \quad : \quad \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\left\{-x^{-\alpha}\right\} & \text{if } x \geq 0 \end{cases}$$
$$\text{for some } \alpha > 0;$$

$$III \quad : \quad \Psi_\alpha(x) = \begin{cases} \exp\left\{-(-x)^\alpha\right\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases}$$
$$\text{for some } \alpha > 0.$$

(4 marks)

SEEN

The necessary and sufficient conditions for the three extreme value distributions are:

$$I \quad : \quad \exists \gamma(t) > 0 \text{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F\left(t + x\gamma(t)\right)}{1 - F(t)} = \exp(-x), \qquad x \in \Re,$$

$$II \quad : \quad w(F) = \infty \text{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \qquad x > 0,$$

$$III \quad : \quad w(F) < \infty \text{ and } \lim_{t \downarrow 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} = x^\alpha, \qquad x > 0.$$

(4 marks)

SEEN

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function say $h(t)$ such that

$$\lim_{t \rightarrow w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} = e^{-x}$$

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for every $x \in (-\infty, \infty)$. But

$$
\begin{aligned}
\lim_{t \to w(F)} \frac{1 - F\left(t + xh(t)\right)}{1 - F(t)} &= \lim_{t \to w(F)} \frac{\left(1 + xh'(t)\right) f\left(t + xh(t)\right)}{f(t)} \\
&= \lim_{t \to w(F)} \frac{\left(1 + xh'(t)\right) g\left(t + xh(t)\right) G^{a-1}\left(t + xh(t)\right) \left[1 - G\left(t + xh(t)\right)\right]^{b-1}}{g(t) G^{a-1}(t) \left[1 - G(t)\right]^{b-1}} \\
&= \lim_{t \to w(F)} \frac{\left(1 + xh'(t)\right) g\left(t + xh(t)\right) \left[1 - G\left(t + xh(t)\right)\right]^{b-1}}{g(t) \left[1 - G(t)\right]^{b-1}} \\
&= \lim_{t \to w(F)} \frac{\left(1 + xh'(t)\right) g\left(t + xh(t)\right)}{g(t)} \frac{\left[1 - G\left(t + xh(t)\right)\right]^{b-1}}{\left[1 - G(t)\right]^{b-1}} \\
&= \lim_{t \to w(F)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} \frac{\left[1 - G\left(t + xh(t)\right)\right]^{b-1}}{\left[1 - G(t)\right]^{b-1}} \\
&= \lim_{t \to w(F)} \left[\frac{1 - G\left(t + xh(t)\right)}{1 - G(t)}\right]^{b} \\
&= \exp(-bx)
\end{aligned}
$$

for every $x \in (-\infty, \infty)$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$
\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left[-\exp(-bx)\right]
$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$
\lim_{t \to \infty} \frac{1 - G\left(tx\right)}{1 - G(t)} = x^{-\beta}
$$

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for every $x > 0$. But

$$\lim_{t \to w(F)} \frac{1 - F(tx)}{1 - F(t)} = \lim_{t \to w(F)} \frac{xf(tx)}{f(t)}$$

$$= \lim_{t \to w(F)} \frac{xg(tx) G^{a-1}(tx) [1 - G(tx)]^{b-1}}{g(t) G^{a-1}(t) [1 - G(t)]^{b-1}}$$

$$= \lim_{t \to w(F)} \frac{xg(tx) [1 - G(tx)]^{b-1}}{g(t) [1 - G(t)]^{b-1}}$$

$$= \lim_{t \to w(F)} \frac{xg(tx)}{g(t)} \frac{[1 - G(tx)]^{b-1}}{[1 - G(t)]^{b-1}}$$

$$= \lim_{t \to w(F)} \frac{1 - G(tx)}{1 - G(t)} \frac{[1 - G(tx)]^{b-1}}{[1 - G(t)]^{b-1}}$$

$$= \lim_{t \to w(F)} \left[\frac{1 - G(t + xh(t))}{1 - G(t)} \right]^{b}$$

$$= x^{-b\beta}$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \le x \right) = \exp\left(-x^{-b\beta}\right)$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t \to 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} = x^{\beta}$$

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for every $x > 0$. But

$$
\begin{aligned}
\lim_{t \downarrow 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F(w(F) - t)} &= \lim_{t \downarrow 0} \frac{x f\left(w(F) - tx\right)}{f(w(F) - t)} \\
&= \lim_{t \downarrow 0} \frac{x g\left(w(F) - tx\right) G^{a-1}\left(w(F) - tx\right)\left[1 - G\left(w(F) - tx\right)\right]^{b-1}}{g(t) G^{a-1}\left(w(F) - t\right)\left[1 - G\left(w(F) - t\right)\right]^{b-1}} \\
&= \lim_{t \downarrow 0} \frac{x g\left(w(F) - tx\right)\left[1 - G\left(w(F) - tx\right)\right]^{b-1}}{g\left(w(F) - t\right)\left[1 - G\left(w(F) - t\right)\right]^{b-1}} \\
&= \lim_{t \downarrow 0} \frac{x g\left(w(F) - tx\right)}{g\left(w(F) - t\right)} \frac{\left[1 - G\left(w(F) - tx\right)\right]^{b-1}}{\left[1 - G\left(w(F) - t\right)\right]^{b-1}} \\
&= \lim_{t \downarrow 0} \frac{1 - G\left(w(F) - tx\right)}{1 - G(w(F) - t)} \frac{\left[1 - G\left(w(F) - tx\right)\right]^{b-1}}{\left[1 - G\left(w(F) - t\right)\right]^{b-1}} \\
&= \lim_{t \downarrow 0} \left[\frac{1 - G\left(w(F) - tx\right)}{1 - G(w(F) - t)}\right]^{b} \\
&= x^{b\beta}
\end{aligned}
$$

for every $x < 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$
\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left(-(-x)^{b\beta}\right)
$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

UNSEEN

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Solutions to Question 3

a) Note that $w(F) = N$ and

$$
\begin{aligned}
\frac{\Pr(X = w(F))}{1 - F(w(F) - 1)} &= \frac{\Pr(X = N)}{1 - F(N - 1)} \\
&= \frac{\Pr(X = N)}{1 - \Pr(X = 1) - \Pr(X = 2) - \cdots - \Pr(X = N - 1)} \\
&= \frac{\frac{1}{N}}{1 - \frac{1}{N} - \frac{1}{N} - \cdots - \frac{1}{N}} \\
&= \frac{\frac{1}{N}}{1 - \frac{N-1}{N}} \\
&= 1.
\end{aligned}
$$

Hence, there can be no non-degenerate limit.

(4 marks)

UNSEEN

b) Note that $w(F) = n$ and

$$
\begin{aligned}
\frac{\Pr(X = w(F))}{1 - F(w(F) - 1)} &= \frac{\Pr(X = n)}{1 - F(n - 1)} \\
&= \frac{\Pr(X = n)}{1 - \Pr(X \leq n - 1)} \\
&= \frac{\Pr(X = n)}{\Pr(X = n)} \\
&= 1.
\end{aligned}
$$

Hence, there can be no non-degenerate limit.

c) Note that $w(F) = b$. The corresponding cdf is

$$
F(x) = \frac{\log x - \log a}{\log b - \log a}.
$$

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Note that

$$\lim_{t\downarrow 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} = \lim_{t\downarrow 0} \frac{1 - F\left(b - tx\right)}{1 - F\left(b - t\right)}$$

$$= \lim_{t\downarrow 0} \frac{1 - \frac{\log(b - tx) - \log a}{\log b - \log a}}{1 - \frac{\log(b - t) - \log a}{\log b - \log a}}$$

$$= \lim_{t\downarrow 0} \frac{\log b - \log(b - tx)}{\log b - \log(b - t)}$$

$$= \lim_{t\downarrow 0} \frac{\frac{x}{b - tx}}{\frac{1}{b - t}}$$

$$= x.$$

So, $F(x)$ belongs to the Weibull domain of attraction.

(4 marks)

UNSEEN

d) Note that $w(F) = \infty$. Then

$$\lim_{t\to\infty} \frac{1 - F\left(t + xg(t)\right)}{1 - F(t)} = \lim_{t\to\infty} \frac{1 - \left\{1 - \exp\left[-(t + xg(t))^2\right]\right\}^a}{1 - \left\{1 - \exp\left[-t^2\right]\right\}^a}$$

$$= \lim_{t\to\infty} \frac{1 - \left\{1 - a\exp\left[-(t + xg(t))^2\right]\right\}}{1 - \left\{1 - a\exp\left[-t^2\right]\right\}}$$

$$= \lim_{t\to\infty} \frac{\exp\left[-(t + xg(t))^2\right]}{\exp\left[-t^2\right]}$$

$$= \lim_{t\to\infty} \exp\left[-2txg(t) - x^2 g^2(t)\right]$$

$$= \exp(-x)$$

if $g(t) = 1/(2t)$. So, $F(x) = [1 - \exp(-x^2)]^a$ belongs to the Gumbel domain of attraction.

(4 marks)

UNSEEN

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e) Note that $w(F) = \infty$. Then

$$
\begin{aligned}
\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} &= \lim_{t \uparrow \infty} \frac{1 - 1 + [1 - \exp(-(tx)^{-1})]^a}{1 - 1 + [1 - \exp(-t^{-1})]^a} \\
&= \lim_{t \uparrow \infty} \frac{[1 - \exp(-(tx)^{-1})]^a}{[1 - \exp(-t^{-1})]^a} \\
&= \lim_{t \uparrow \infty} \left[\frac{1 - \exp(-(tx)^{-1})}{1 - \exp(-t^{-1})} \right]^a \\
&= \lim_{t \uparrow \infty} \left[\frac{1 - 1 + (tx)^{-1}}{1 - 1 + t^{-1}} \right]^a \\
&= x^{-a}.
\end{aligned}
$$

. So, the cdf $F(x) = 1 - [1 - \exp(-x^{-1})]^a$ belongs to the Fréchet domain of attraction.

(4 marks)

UNSEEN

Solutions to Question 4

(a) If X is an absolutely continuous random variable with cdf $F(\cdot)$ then

$$\mathrm{VaR}_p(X) = F^{-1}(p)$$

and

$$\mathrm{ES}_p(X) = \frac{1}{p}\int_0^p F^{-1}(v)dv.$$

<div align="right">(2 marks)</div>

SEEN

(b) (i) The corresponding cdf is

$$F(x) = \int_{-a}^{x} \frac{3y^2}{2a^3}dy = \frac{3}{2a^3}\left[\frac{y^3}{3}\right]_{-a}^{x} = \frac{3}{2a^3}\left[\frac{x^3 + a^3}{3}\right] = \frac{x^3 + a^3}{2a^3}$$

for $-a < x < a$;

<div align="right">(2 marks)</div>

UNSEEN

(b) (ii) Inverting

$$F(x) = \frac{x^3 + a^3}{2a^3} = p,$$

we obtain $\mathrm{VaR}_p(X) = (2p - 1)^{1/3}a$.

<div align="right">(1 marks)</div>

UNSEEN

(b) (iii) The expected shortfall is

$$\mathrm{ES}_p(X) = \frac{1}{p}\int_0^p (2v - 1)^{1/3}dv = \frac{1}{p}\left[\frac{3(2v - 1)^{4/3}}{8}\right]_0^p = \frac{3a}{8p}\left[(2p - 1)^{4/3} - 1\right].$$

<div align="right">(2 marks)</div>

UNSEEN

c) (i) The likelihood function of a is

$$
\begin{aligned}
L(a) &= \frac{3^n}{2^n a^{3n}} \prod_{i=1}^{n} \left[X_i^2 I\{-a < X_i < a\} \right] \\
&= \frac{3^n}{2^n a^{3n}} \left(\prod_{i=1}^{n} X_i \right)^2 \prod_{i=1}^{n} I\{-a < X_i < a\} \\
&= \frac{3^n}{2^n a^{3n}} \left(\prod_{i=1}^{n} X_i \right)^2 I\{\max(X_1, \ldots, X_n) < a, \min(X_1, \ldots, X_n) > -a\} \\
&= \frac{3^n}{2^n a^{3n}} \left(\prod_{i=1}^{n} X_i \right)^2 I\{a > \max(X_1, \ldots, X_n), a > -\min(X_1, \ldots, X_n)\} \\
&= \frac{3^n}{2^n a^{3n}} \left(\prod_{i=1}^{n} X_i \right)^2 I\{a > \max[\max(X_1, \ldots, X_n), -\min(X_1, \ldots, X_n)]\}.
\end{aligned}
$$

(3 marks)

UNSEEN

c) (ii) Note that $L(a)$ is a decreasing function over a and $a > \max[\max(X_1, \ldots, X_n), -\min(X_1, \ldots, X_n)]$.
Hence, the mle of a is $\max[\max(X_1, \ldots, X_n), -\min(X_1, \ldots, X_n)]$.

(1 marks)

UNSEEN

c) (iii) The mles of VaR and ES are $\widehat{\mathrm{VaR}}_p(X) = (2p-1)^{1/3}\widehat{a}$ and

$$
\widehat{\mathrm{ES}}_p(X) = \frac{3\widehat{a}}{8p} \left[(2p-1)^{4/3} - 1 \right],
$$

where $\widehat{a} = \max[\max(X_1, \ldots, X_n), -\min(X_1, \ldots, X_n)]$.

(2 marks)

UNSEEN

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c (iv) Let $Z = \max\left[\max\left(X_1,\ldots,X_n\right), -\min\left(X_1,\ldots,X_n\right)\right]$. The cdf of Z is

$$
\begin{aligned}
F_Z(z) &= \Pr\left(\max\left[\max\left(X_1,\ldots,X_n\right), -\min\left(X_1,\ldots,X_n\right)\right] \le z\right)\\
&= \Pr\left(\max\left(X_1,\ldots,X_n\right) \le z, -\min\left(X_1,\ldots,X_n\right) \le z\right)\\
&= \Pr\left(\max\left(X_1,\ldots,X_n\right) \le z, \min\left(X_1,\ldots,X_n\right) \ge -z\right)\\
&= \Pr\left(X_1 \le z, \ldots, X_n \le z, X_1 \ge -z, \ldots, X_n \ge -z\right)\\
&= \Pr\left(-z \le X_1 \le z, \ldots, -z \le X_n \le z\right)\\
&= \Pr^n\left(-z \le X \le z\right)\\
&= \left[\Pr\left(X \le z\right) - \Pr\left(X \le -z\right)\right]^n\\
&= \left[\frac{z^3}{a^3}\right]^n
\end{aligned}
$$

for $z > 0$. The corresponding pdf is

$$
f_Z(z) = 3na^{-3n}z^{3n-1}
$$

for $z > 0$.

(3 marks)

UNSEEN

c (v) The expected value of Z is

$$
E(Z) = 3na^{-3n}\int_0^a z^{3n}dz = 3na^{-3n}\left[\frac{z^{3n+1}}{3n+1}\right]_0^a dz = 3na^{-3n}\frac{a^{3n+1}}{3n+1} = \frac{3na}{3n+1}.
$$

Hence, $\widehat{a}$ is biased for a.

(2 marks)

UNSEEN

c (vi) Since

$$
\text{Bias}\left[\widehat{\text{VaR}}_p(X)\right] = (2p-1)^{1/3}\text{Bias}\left[\widehat{a}\right],
$$

we see that $\widehat{\text{VaR}}_p(X)$ is biased for $\text{VaR}_p(X)$.

(1 marks)

UNSEEN

c (vii) Since

$$
\text{Bias}\left[\widehat{\text{ES}}_p(X)\right] = \frac{3}{8p}\left[(2p-1)^{4/3}-1\right]\text{Bias}\left[\widehat{a}\right],
$$

we see that $\widehat{\mathrm{ES}}_p(X)$ is biased for $\mathrm{ES}_p(X)$.

UNSEEN

Solutions to Question 5

a) Note that

$$\frac{\partial \overline{F}}{\partial x_1} = -\frac{\alpha}{\theta} \left(1 + \frac{1}{\theta} \sum_{i=1}^{k} x_i \right)^{-\alpha-1},$$

$$\frac{\partial^2 \overline{F}}{\partial x_1 \partial x_2} = \frac{\alpha(\alpha+1)}{\theta^2} \left(1 + \frac{1}{\theta} \sum_{i=1}^{k} x_i \right)^{-\alpha-2},$$

$$\frac{\partial^3 \overline{F}}{\partial x_1 \partial x_2 \partial x_3} = -\frac{\alpha(\alpha+1)(\alpha+3)}{\theta^3} \left(1 + \frac{1}{\theta} \sum_{i=1}^{k} x_i \right)^{-\alpha-3},$$

and so on. In general,

$$f(x_1, x_2, \ldots, x_k) = \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\theta^k} \left(1 + \frac{1}{\theta} \sum_{i=1}^{k} x_i \right)^{-\alpha-k},$$

as required.

(4 marks)

UNSEEN

b) The pdf of S is

$$
\begin{aligned}
f_S(s) &= \int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{k-1}} f(x_1, x_2, \ldots, t - x_1 - x_2 - \cdots - x_{k-1}) \, dx_k \cdots dx_2 dx_1 \\
&= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\theta^k} \int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{k-1}} \left(1 + \frac{1}{\theta} \sum_{i=1}^{k} x_i \right)^{-\alpha-k} dx_k \cdots dx_2 dx_1 \\
&= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\theta^k} \int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{k-1}} \left(1 + \frac{s}{\theta} \right)^{-\alpha-k} dx_k \cdots dx_2 dx_1 \\
&= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\theta^k} \left(1 + \frac{s}{\theta} \right)^{-\alpha-k} \int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{k-1}} dx_k \cdots dx_2 dx_1 \\
&= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k! \theta^k} s^k \left(1 + \frac{s}{\theta} \right)^{-\alpha-k},
\end{aligned}
$$

as required.

(4 marks)

UNSEEN

c) The cdf of S is

$$
\begin{aligned}
F_S(s) &= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!\theta^k} \int_0^s x^k \left(1+\frac{x}{\theta}\right)^{-\alpha-k} dx \\
&= \frac{\theta\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!} \int_1^{\theta/(s+\theta)} y^{\alpha-2}(1-y)^k dy \\
&= \frac{\theta\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!} \int_0^{s/(s+\theta)} (1-y)^{\alpha-2} y^k dy \\
&= \frac{\theta\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!} B_{s/(s+\theta)}(k+1, \alpha-1),
\end{aligned}
$$

as required.

(4 marks)

UNSEEN

d) The nth moment of S is

$$
\begin{aligned}
E(S^n) &= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!\theta^k} \int_0^\infty x^{n+k} \left(1+\frac{x}{\theta}\right)^{-\alpha-k} dx \\
&= \frac{\theta^{n+1}\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!} \int_0^1 (1-y)^{n+k} y^{\alpha-n-2} dy \\
&= \frac{\theta^{n+1}\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!} B(\alpha-n-1, n+k+1).
\end{aligned}
$$

(4 marks)

UNSEEN

e) The joint likelihood function is

$$
L(\alpha,\theta) = \frac{\alpha^n(\alpha+1)^n\cdots(\alpha+k-1)^n}{(k!)^n \theta^{kn}} \left(\prod_{i=1}^n s_i\right)^k \left[\prod_{i=1}^n \left(1+\frac{s_i}{\theta}\right)\right]^{-\alpha-k}.
$$

Its log is The joint likelihood function is

$$
\log L(\alpha,\theta) = \log\left[\frac{\alpha^n(\alpha+1)^n\cdots(\alpha+k-1)^n}{(k!)^n \theta^{kn}}\right] + k\sum_{i=1}^n \log s_i - (\alpha+k)\sum_{i=1}^n \log\left(1+\frac{s_i}{\theta}\right).
$$

The partial derivatives of this with respect to α and θ are

$$
\frac{\partial \log L(\alpha,\theta)}{\partial \alpha} = \frac{n}{\alpha} + \frac{n}{\alpha+1} + \cdots + \frac{n}{\alpha+k-1} - \sum_{i=1}^n \log\left(1+\frac{s_i}{\theta}\right)
$$

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and

$$\frac{\partial \log L\left(\alpha, \theta\right)}{\partial \theta} = -\frac{kn}{\theta} + \frac{\alpha + k}{\theta^2} \sum_{i=1}^{n} s_i \left(1 + \frac{s_i}{\theta}\right)^{-1}.$$

The maximum likelihood estimates of α and θ are the simultaneous solutions of

$$\frac{n}{\alpha} + \frac{n}{\alpha + 1} + \cdots + \frac{n}{\alpha + k - 1} = \sum_{i=1}^{n} \log \left(1 + \frac{s_i}{\theta}\right)$$

and

$$\frac{kn}{\theta} = \frac{\alpha + k}{\theta^2} \sum_{i=1}^{n} s_i \left(1 + \frac{s_i}{\theta}\right)^{-1}.$$

(4 marks)

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