SOLUTIONS TO MATH68181 EXTREME VALUES AND FINANCIAL RISK EXAM

Solutions to Question A1

a) The marginal cdfs of

$$F_{X,Y}(x,y) = \left[1 + \exp(-x) + \exp(-y) + (1-\alpha)\exp(-x-y)\right]^{-1}$$

are

$$F_X(x) = F_{X,Y}(x,\infty) = \left[1 + \exp(-x)\right]^{-1}$$

and

$$F_Y(y) = F_{X,Y}(\infty,y) = \left[1 + \exp(-y)\right]^{-1}.$$

(3 marks)

UNSEEN

b) F_X belongs to the Gumbel max domain of attraction since

$$\begin{aligned}
\lim_{t \to w(F_X)} \frac{1 - F_X(t+x)}{1 - F_X(t)} &= \lim_{t \to \infty} \frac{1 - \left[1 + e^{-t-x}\right]^{-1}}{1 - \left[1 + e^{-t}\right]^{-1}} \\
&= \lim_{t \to \infty} \frac{1 - \left[1 - e^{-t-x}\right]}{1 - \left[1 - e^{-t}\right]} \\
&= e^{-x}.
\end{aligned}$$

(3 marks)

c) F_Y belongs to the Gumbel max domain of attraction since

$$\begin{aligned}
\lim_{t \to w(F_Y)} \frac{1 - F_Y(t+x)}{1 - F_Y(t)} &= \lim_{t \to \infty} \frac{1 - \left[1 + e^{-t-y}\right]^{-1}}{1 - \left[1 + e^{-t}\right]^{-1}} \\
&= \lim_{t \to \infty} \frac{1 - \left[1 - e^{-t-y}\right]}{1 - \left[1 - e^{-t}\right]} \\
&= e^{-y}.
\end{aligned}$$

(2 marks)

SEEN

d) Use the rule $a_n = \gamma^{-1}\left(F_X^{-1}\left(1 - n^{-1}\right)\right)$ and $b_n = F_X^{-1}\left(1 - n^{-1}\right)$. From part (b), $\gamma(t) = 1$. Inverting

$$F_X(x) = \left[1 + \exp(-x)\right]^{-1} = 1 - n^{-1},$$

we obtain

$$F_X^{-1}\left(1-n^{-1}\right) = \log(n-1).$$

(3 marks)

UNSEEN

e) Use the rule $c_n = \gamma^{-1}\left(F_Y^{-1}\left(1-n^{-1}\right)\right)$ and $d_n = F_Y^{-1}\left(1-n^{-1}\right)$. From part (c), $\gamma(t) = 1$. Inverting

$$F_Y(y) = \left[1+\exp(-y)\right]^{-1} = 1-n^{-1},$$

we obtain

$$F_Y^{-1}\left(1-n^{-1}\right) = \log(n-1).$$

(2 marks)

UNSEEN

g) The limiting cdf is

$$\begin{aligned}
& \lim_{n\to\infty} F_{X,Y}\left(a_n x + b_n, c_n y + d_n\right) \\
= & \lim_{n\to\infty} \left[1+\exp\left(-x-\log(n-1)\right)+\exp\left(-y-\log(n-1)\right)+(1-\alpha)\exp\left(-x-y-2\log(n-1)\right)\right]^{-n} \\
= & \lim_{n\to\infty} \left[1+\frac{\exp\left(-x\right)}{n-1}+\frac{\exp\left(-y\right)}{n-1}+(1-\alpha)\frac{\exp\left(-x-y\right)}{(n-1)^2}\right]^{-n} \\
= & \exp\left\{-\exp(-x)-\exp(-y)\right\}.
\end{aligned}$$

(5 marks)

UNSEEN

h) Yes, the extremes are completely independent since

$$\begin{aligned}
\lim_{n\to\infty} F_{X,Y}\left(a_n x + b_n, c_n y + d_n\right) &= \exp\left\{-\exp(-x)-\exp(-y)\right\} \\
&= \lim_{n\to\infty} F_X\left(a_n x + b_n\right) \lim_{n\to\infty} F_Y\left(c_n y + d_n\right).
\end{aligned}$$

(2 marks)

UNSEEN

Solutions to Question A2

$C(u_1, u_2)$ is a valid copula if

$$C(u, 0) = 0,$$

$$C(0, u) = 0,$$

$$C(1, u) = u,$$

$$C(u, 1) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) \geq 0.$$

(4 marks)

SEEN

a) for the copula defined by $C(u_1, u_2) = \left[\alpha \left(\min(u_1, u_2)\right)^m + (1-\alpha) u_1^m u_2^m\right]^{1/m}$, we have

$$C(u, 0) = [\alpha \cdot 0 + (1-\alpha) \cdot 0]^{1/m} = 0,$$

$$C(0, u) = [\alpha \cdot 0 + (1-\alpha) \cdot 0]^{1/m} = 0,$$

$$C(1, u) = [\alpha u^m + (1-\alpha) u^m]^{1/m} = u,$$

$$C(u, 1) = [\alpha u^m + (1-\alpha) u^m]^{1/m} = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \begin{cases} [\alpha + (1-\alpha) u_2^m]^{1/m}, & \text{if } u_1 \leq u_2, \\ (1-\alpha) u_1^{m-1} u_2 \left[\alpha + (1-\alpha) u_2^m\right]^{1/m-1}, & \text{if } u_2 \leq u_1 \end{cases} \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \begin{cases} (1-\alpha) u_1 u_2^{m-1} \left[\alpha + (1-\alpha) u_2^m\right]^{1/m-1}, & \text{if } u_1 \leq u_2, \\ [\alpha + (1-\alpha) u_2^m]^{1/m}, & \text{if } u_2 \leq u_1 \end{cases} \geq 0.$$

(4 marks)

b) for the copula defined by $C(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$, we have

$$C(u, 0) = \max(u + 0 - 1, 0) = 0,$$

$$C(0, u) = \max(0 + u - 1, 0) = 0,$$

$$C(1, u) = \max(1 + u - 1, 0) = u,$$

$$C(u, 1) = \max(u + 1 - 1, 0) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \left\{ \begin{array}{ll} 1, & \text{if } u_1 + u_2 \geq 1, \\ 0, & \text{if } u_1 + u_2 < 1 \end{array} \right. \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \left\{ \begin{array}{ll} 1, & \text{if } u_1 + u_2 \geq 1, \\ 0, & \text{if } u_1 + u_2 < 1 \end{array} \right. \geq 0.$$

(4 marks)

c) for the copula defined by $C(u_1, u_2) = \min\left(u_1^a, u_2^b\right) \min\left(u_1^{1-a}, u_2^{1-b}\right)$, we have

$$C(u, 0) = \min(u^a, 0) \min\left(u^{1-a}, 0\right) = 0,$$

$$C(0, u) = \min\left(0, u^b\right) \min\left(0, u^{1-b}\right) = 0,$$

$$C(1, u) = \min\left(1, u^b\right) \min\left(1, u^{1-b}\right) = u,$$

$$C(u, 1) = \min(u^a, 1) \min\left(u^{1-a}, 1\right) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \left\{ \begin{array}{ll} 1, & \text{if } u_1^a \leq u_2^b \text{ and } u_1^{1-a} \leq u_2^{1-b}, \\ a u_1^{a-1} u_2^{1-b}, & \text{if } u_1^a \leq u_2^b \text{ and } u_1^{1-a} > u_2^{1-b}, \\ (1-a) u_1^{-a} u_2^b, & \text{if } u_1^a > u_2^b \text{ and } u_1^{1-a} \leq u_2^{1-b}, \\ 0, & \text{if } u_1^a > u_2^b \text{ and } u_1^{1-a} > u_2^{1-b} \end{array} \right. \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \left\{ \begin{array}{ll} 0, & \text{if } u_1^a \leq u_2^b \text{ and } u_1^{1-a} \leq u_2^{1-b}, \\ (1-b) u_1^a u_2^{-b}, & \text{if } u_1^a \leq u_2^b \text{ and } u_1^{1-a} > u_2^{1-b}, \\ b u_1^{1-a} u_2^{b-1}, & \text{if } u_1^a > u_2^b \text{ and } u_1^{1-a} \leq u_2^{1-b}, \\ 1, & \text{if } u_1^a > u_2^b \text{ and } u_1^{1-a} > u_2^{1-b} \end{array} \right. \geq 0.$$

(4 marks)

d) for the copula defined by $C\left(u_1,u_2\right)=\exp\left\{-\left[\left(-\log u_1\right)^\theta+\left(-\log u_2\right)^\theta\right]^{1/\theta}\right\}$, we have

$$C\left(u,0\right)=\exp\left\{-\left[\left(-\log u\right)^\theta+\left(\infty\right)^\theta\right]^{1/\theta}\right\}=0,$$

$$C\left(0,u\right)=\exp\left\{-\left[\left(\infty\right)^\theta+\left(-\log u\right)^\theta\right]^{1/\theta}\right\}=0,$$

$$C\left(1,u\right)=\exp\left\{-\left[0+\left(-\log u\right)^\theta\right]^{1/\theta}\right\}=u,$$

$$C\left(u,1\right)=\exp\left\{-\left[\left(-\log u\right)^\theta+0\right]^{1/\theta}\right\}=u,$$

$$\frac{\partial}{\partial u_1}C\left(u_1,u_2\right)=u_1^{-1}\left(-\log u_1\right)^{\theta-1}\left[\left(-\log u_1\right)^\theta+\left(-\log u_2\right)^\theta\right]^{1/\theta-1}$$
$$\cdot\exp\left\{-\left[\left(-\log u_1\right)^\theta+\left(-\log u_2\right)^\theta\right]^{1/\theta}\right\}\geq 0$$

and

$$\frac{\partial}{\partial u_1}C\left(u_1,u_2\right)=u_2^{-1}\left(-\log u_2\right)^{\theta-1}\left[\left(-\log u_1\right)^\theta+\left(-\log u_2\right)^\theta\right]^{1/\theta-1}$$
$$\cdot\exp\left\{-\left[\left(-\log u_1\right)^\theta+\left(-\log u_2\right)^\theta\right]^{1/\theta}\right\}\geq 0.$$

(4 marks)

Solutions to Question A3

a) We can write

$$\overline{F}(x,y) = \exp\left\{-(x+y)\left[1-(\theta+\phi)\frac{y}{x+y}+\frac{\theta y^2}{(x+y)^2}+\frac{\phi y^3}{(x+y)^3}\right]\right\}$$

for $x > 0$, $y > 0$, $\theta \geq 0$, $\phi \geq 0$, $\theta + 3\phi \geq 0$, $\theta + \phi \leq 1$ and $\theta + 2\phi \leq 1$. This is in the form of

$$\overline{F}(x,y) = \exp\left[-(x+y)A\left(\frac{y}{x+y}\right)\right]$$

with $A(w) = 1 - (\theta+\phi)w + \theta w^2 + \phi w^3$.

We now check the conditions for $A(\cdot)$. Clearly, $A(0) = 1$ and $A(1) = 1$.

Also $A(t) \geq 0$ since $\theta + \phi \leq 1$ implies $1 - (\theta+\phi)w \leq 1$ for all w.

Also $A(w) \leq 1$ since

$$\begin{aligned} & A(w) \leq 1 \\ \Leftrightarrow\ & 1-(\theta+\phi)w+\theta w^2+\phi w^3 \leq 1 \\ \Leftrightarrow\ & \theta\left(w^2-w\right)+\phi\left(w^3-w\right) \leq 0 \\ \Leftrightarrow\ & \theta w\left(w-1\right)+\phi w\left(w^2-1\right) \leq 0 \\ \Leftrightarrow\ & \left(\theta+\phi+\phi w\right) w\left(w-1\right) \leq 0. \end{aligned}$$

Note that $\max(w, 1-w) = w$ if $w \in [1/2, 1]$. So, for $w \in [1/2, 1]$,

$$\begin{aligned} & A(w) \geq \max(w, 1-w) \\ \Leftrightarrow\ & A(w) \geq w \\ \Leftrightarrow\ & 1-(\theta+\phi+1)w+\theta w^2+\phi w^3 \geq 0. \end{aligned}$$

Let $g(w) = 1-(\theta+\phi+1)w+\theta w^2+\phi w^3$. Note that $g^{'}(w) = 2\theta w + 3\phi w^2 - \theta - \phi - 1 \leq 0$ for all $w \in [1/2, 1]$. But $g(1) = 0 \geq 0$, so $A(w) \geq \max(w, 1-w)$ for all $w \in [1/2, 1]$.

Note that $\max(w, 1-w) = 1-w$ if $w \in [0, 1/2]$. So, for $w \in [0, 1/2]$,

$$\begin{aligned} & A(w) \geq \max(w, 1-w) \\ \Leftrightarrow\ & A(w) \geq 1-w \\ \Leftrightarrow\ & (1-\theta-\phi)w+\theta w^2+\phi w^3 \geq 0, \end{aligned}$$

which holds since $\theta + \phi \leq 1$.

$A(\cdot)$ is convex since

$$A^{'}(w) = 2\theta w + 3\phi w^2 - \theta - \phi$$

and

$$A''(t) = 2\theta + 6\phi w \geq 0$$

for all w since $\theta + 3\phi \geq 0$.

(6 marks)

UNSEEN

b) the joint cdf is

$$F(x,y) = 1 - \exp(-x) - \exp(-y) + \exp\left\{-(x+y)\left[1 - (\theta+\phi)\frac{y}{x+y} + \frac{\theta y^2}{(x+y)^2} + \frac{\phi y^3}{(x+y)^3}\right]\right\}.$$

(2 marks)

UNSEEN

c) the derivative of joint cdf with respect to x is

$$\frac{\partial F(x,y)}{\partial x} = \exp(-x) + \overline{F}(x,y)\left[\frac{\theta y^2}{(x+y)^2} + \frac{2\phi y^3}{(x+y)^3} - 1\right],$$

so the conditional cdf if Y given $X = x$ is

$$F(y|x) = 1 + \exp(x)\overline{F}(x,y)\left[\frac{\theta y^2}{(x+y)^2} + \frac{2\phi y^3}{(x+y)^3} - 1\right].$$

(4 marks)

UNSEEN

d) the derivative of joint cdf with respect to y is

$$\frac{\partial F(x,y)}{\partial y} = \exp(-y) + \overline{F}(x,y)\left[\frac{2\phi y^3}{(x+y)^3} + \frac{(\theta - 3\phi)y^2}{(x+y)^2} - \frac{2\theta y}{x+y} + \theta + \phi - 1\right],$$

so the conditional cdf if X given $Y = y$ is

$$F(x|y) = 1 + \exp(y)\overline{F}(x,y)\left[\frac{2\phi y^3}{(x+y)^3} + \frac{(\theta - 3\phi)y^2}{(x+y)^2} - \frac{2\theta y}{x+y} + \theta + \phi - 1\right].$$

(4 marks)

UNSEEN

e) the derivative of joint cdf with respect to x and y is

$$
\begin{aligned}
f(x,y) &= \frac{\partial F(x,y)}{\partial x \partial y} \\
= \quad & \overline{F}(x,y)\left[\frac{\theta y^2}{(x+y)^2}+\frac{2\phi y^3}{(x+y)^3}-1\right]\left[\frac{2\phi y^3}{(x+y)^3}+\frac{(\theta-3\phi)y^2}{(x+y)^2}-\frac{2\theta y}{x+y}+\theta+\phi-1\right] \\
& +\overline{F}(x,y)\left[\frac{2\theta y}{(x+y)^2}+\frac{2(3\phi-\theta)y^2}{(x+y)^3}-\frac{6\theta y^3}{(x+y)^4}\right].
\end{aligned}
$$

(4 marks)

UNSEEN

Solutions to Question B1

a) The mgf of X_i is

$$M_{X_i}(t) = \int_0^\infty \lambda \exp\left[-(\lambda - t)x\right] dx = \left[\frac{\lambda}{t-\lambda} \exp\left[-(\lambda - t)x\right]\right]_0^\infty = 0 - \frac{\lambda}{t-\lambda} = \frac{\lambda}{\lambda - t}.$$

(2 marks)

SEEN

b) The mgf of T conditional on $N = n$ is

$$\begin{aligned} M_{T|N=n}(t) &= E\left\{\exp\left[t\left(X_1 + \cdots + X_n\right)\right]\right\} \\ &= E\left\{\exp\left[tX_1\right] \cdots \exp\left[X_n\right]\right\} \\ &= E\left\{\exp\left[tX_1\right]\right\} \cdots E\left\{\exp\left[X_n\right]\right\} \\ &= M_{X_1}(t) \cdots M_{X_n}(t) \\ &= \left(\frac{\lambda}{\lambda - t}\right)^n \end{aligned}$$

(3 marks)

UNSEEN

c) $\left(\frac{\lambda}{\lambda - t}\right)^n$ is the mgf of a gamma random variable with parameters λ and n. So, the conditional distribution of T is gamma with parameters λ and n.

(3 marks)

UNSEEN

d) The conditional pdf of T is

$$f_{T|N=n}(x) = \frac{\lambda^n x^{n-1} \exp(-\lambda x)}{\Gamma(n)}.$$

So, the unconditional pdf of T is

$$\begin{aligned} f_T(x) &= \sum_{n=1}^{\infty} f_{T|N=n}(x)\theta(1-\theta)^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{\lambda^n x^{n-1}\exp(-\lambda x)}{\Gamma(n)}\theta(1-\theta)^{n-1} \\ &= \theta\lambda\exp(-\lambda x)\sum_{n=1}^{\infty} \frac{(\lambda(1-\theta)x)^{n-1}}{(n-1)!} \\ &= \theta\lambda\exp(-\lambda x)\exp\left[\lambda x(1-\theta)\right] \\ &= \theta\lambda\exp(-\theta\lambda x), \end{aligned}$$

an Exponential pdf with parameter $\theta\lambda$.

(3 marks)

UNSEEN

e) The mean is $1/(\theta\lambda)$ and the variance $1/(\theta\lambda)^2$.

(3 marks)

UNSEEN

f) The unconditional cdf of T is

$$F_T(x) = 1 - \exp(-\theta\lambda x).$$

Inverting

$$1 - \exp(-\theta\lambda x) = p,$$

we obtain

$$\mathrm{VaR}_p(T) = -\frac{1}{\theta\lambda}\log(1-p).$$

(3 marks)

UNSEEN

g) The expected shortfall T is

$$\begin{aligned}
\mathrm{ES}_p(T) &= -\frac{1}{\theta\lambda p}\int_0^p \log(1-t)dt \\
&= -\frac{1}{\theta\lambda p}\left\{[t\log(1-t)]_0^p + \int_0^p \frac{t}{1-t}dt\right\} \\
&= -\frac{1}{\theta\lambda p}\left\{p\log(1-p) + \int_0^p \frac{t-1+1}{1-t}dt\right\} \\
&= -\frac{1}{\theta\lambda p}\left\{p\log(1-p) - p + \int_0^p \frac{1}{1-t}dt\right\} \\
&= -\frac{1}{\theta\lambda p}\left\{p\log(1-p) - p + [-\log(1-t)]_0^p\right\} \\
&= -\frac{1}{\theta\lambda p}\left\{p\log(1-p) - p - \log(1-p)\right\}.
\end{aligned}$$

(3 marks)

UNSEEN

Solutions to Question B2

If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cdf of a normalized version of M_n converges to G, i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n\left(a_n x + b_n\right) \to G(x) \tag{1}$$

as $n \to \infty$ then G must be of the same type as (cdfs G and G^* are of the same type if $G^*(x) = G(ax+b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$\begin{aligned}
I &: \quad \Lambda(x) = \exp\left\{-\exp(-x)\right\}, \qquad x \in \Re; \\
II &: \quad \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\left\{-x^{-\alpha}\right\} & \text{if } x \geq 0 \end{cases} \\
&\quad\ \text{for some } \alpha > 0; \\
III &: \quad \Psi_\alpha(x) = \begin{cases} \exp\left\{-(-x)^\alpha\right\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \\
&\quad\ \text{for some } \alpha > 0.
\end{aligned}$$

(6 marks)

SEEN

The necessary and sufficient conditions for the three extreme value distributions are:

$$\begin{aligned}
I &: \quad \exists \gamma(t) > 0 \text{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F\left(t + x\gamma(t)\right)}{1 - F(t)} = \exp(-x), \qquad x \in \Re, \\
II &: \quad w(F) = \infty \text{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \qquad x > 0, \\
III &: \quad w(F) < \infty \text{ and } \lim_{t \downarrow 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} = x^\alpha, \qquad x > 0.
\end{aligned}$$

(6 marks)

SEEN

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function say $h(t)$ such that

$$\lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} = e^{-x}$$

for every $x \in (-\infty, \infty)$. But

$$\begin{aligned}
\lim_{t \to W(F)} \frac{1 - F(t + xh(t))}{1 - F(t)} &= \lim_{t \to w(F)} \left\{ \frac{1 - \{1 - [1 - G(t + xh(t))]^a\}^b}{1 - \{1 - [1 - G(t)]^a\}^b} \right\}^\theta \\
&= \lim_{t \to w(F)} \left\{ \frac{1 - \{1 - b[1 - G(t + xh(t))]^a\}}{1 - \{1 - b[1 - G(t)]^a\}} \right\}^\theta \\
&= \lim_{t \to w(F)} \left\{ \frac{b[1 - G(t + xh(t))]^a}{b[1 - G(t)]^a} \right\}^\theta \\
&= \lim_{t \to w(F)} \left\{ \frac{[1 - G(t + xh(t))]^a}{[1 - G(t)]^a} \right\}^\theta \\
&= \lim_{t \to w(G)} \left\{ \frac{1 - G(t + xh(t))}{1 - G(t)} \right\}^{a\theta} \\
&= e^{-a\theta x}
\end{aligned}$$

for every $x \in (-\infty, \infty)$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \le x \right) = \exp\left[-\exp(-a\theta x) \right]$$

for some suitable norming constants $a_n > 0$ and b_n.

(3 marks)

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t \to \infty} \frac{1 - G(tx)}{1 - G(t)} = x^{-\beta}$$

for every $x > 0$. But

$$\begin{aligned}
\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} &= \lim_{t \to \infty} \left\{ \frac{1 - \{1 - [1 - G(tx)]^a\}^b}{1 - \{1 - [1 - G(t)]^a\}^b} \right\}^\theta \\
&= \lim_{t \to \infty} \left\{ \frac{1 - \{1 - b[1 - G(tx)]^a\}}{1 - \{1 - b[1 - G(t)]^a\}} \right\}^\theta \\
&= \lim_{t \to \infty} \left\{ \frac{b[1 - G(tx)]^a}{b[1 - G(t)]^a} \right\}^\theta \\
&= \lim_{t \to \infty} \left\{ \frac{[1 - G(tx)]^a}{[1 - G(t)]^a} \right\}^\theta \\
&= \lim_{t \to \infty} \left\{ \frac{1 - G(tx)}{1 - G(t)} \right\}^{a\theta} \\
&= x^{\beta a\theta}
\end{aligned}$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \le x\right) = \exp\left(-x^{-\beta a\theta}\right)$$

for some suitable norming constants $a_n > 0$ and b_n.

(3 marks)

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t\to 0} \frac{1 - G\left(w(G) - tx\right)}{1 - G\left(w(G) - t\right)} = x^{\beta}$$

for every $x > 0$. But But

$$\begin{aligned}
\lim_{t\downarrow 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} &= \lim_{t\downarrow 0} \left\{\frac{1 - \left\{1 - \left[1 - G\left(w(F) - tx\right)\right]^a\right\}^b}{1 - \left\{1 - \left[1 - G\left(w(F) - t\right)\right]^a\right\}^b}\right\}^{\theta} \\
&= \lim_{t\downarrow 0} \left\{\frac{1 - \left\{1 - b\left[1 - G\left(w(F) - tx\right)\right]^a\right\}}{1 - \left\{1 - b\left[1 - G\left(w(F) - t\right)\right]^a\right\}}\right\}^{\theta} \\
&= \lim_{t\downarrow 0} \left\{\frac{b\left[1 - G\left(w(F) - tx\right)\right]^a}{b\left[1 - G\left(w(F) - t\right)\right]^a}\right\}^{\theta} \\
&= \lim_{t\downarrow 0} \left\{\frac{\left[1 - G\left(w(F) - tx\right)\right]^a}{\left[1 - G\left(w(F) - t\right)\right]^a}\right\}^{\theta} \\
&= \lim_{t\to\infty} \left\{\frac{1 - G\left(w(F) - tx\right)}{1 - G\left(w(F) - t\right)}\right\}^{a\theta} \\
&= x^{\beta a\theta}
\end{aligned}$$

for every $x < 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \le x\right) = \exp\left(-(-x)^{\beta a\theta}\right)$$

for some suitable norming constants $a_n > 0$ and b_n.

(2 marks)

UNSEEN

Solutions to Question B3

a) Note that $w(F)=1$. Then

$$\begin{aligned}\lim_{t\downarrow 0}\frac{1-F(1-tx)}{1-F(1-t)} &= \lim_{t\downarrow 0}\frac{xf(1-tx)}{f(1-t)}\\ &= \lim_{t\downarrow 0}\frac{x(1-tx)^{\alpha-1}(tx)^{\beta-1}}{(1-t)^{\alpha-1}t^{\beta-1}}\\ &= x^{\beta}.\end{aligned}$$

So, $f(x)=Cx^{\alpha-1}(1-x)^{\beta-1}$ belongs to the Weibull domain of attraction.

(4 marks)

UNSEEN

b) Note that

$$F(k)=\begin{cases}1/2, & \text{if } k=-1,0,\\ 1, & \text{if } k\geq 1,\\ 0, & \text{otherwise}\end{cases}$$

and

$$1-F(k-1)=\begin{cases}1, & \text{if } k\leq 0,\\ 1/2, & \text{if } k=1,2,\\ 0, & \text{otherwise.}\end{cases}$$

So,

$$\frac{p(k)}{1-F(k-1)}=\begin{cases}1/2, & \text{if } k=-1,\\ 1, & \text{if } k=1,\\ 0, & \text{if } k\leq 2,\ k\neq -1,1,\\ \text{undefined}, & \text{otherwise.}\end{cases}$$

Hence, there can be no non-degenerate limit.

(4 marks)

UNSEEN

c) Note that $w(F)=\infty$. Then

$$\begin{aligned}\lim_{t\to\infty}\frac{1-F(tx)}{1-F(t)} &= \lim_{t\to\infty}\frac{xf(tx)}{f(t)}\\ &= \lim_{t\to\infty}\frac{x\left(1+t^2x^2\right)^{-1}}{\left(1+t^2\right)^{-1}}\\ &= x^{-1}.\end{aligned}$$

So, $f(x) = \pi^{-1}\left(1+x^2\right)^{-1}$ belongs to the Fréchet domain of attraction.

(4 marks)

UNSEEN

d) Note that $w(F) = \infty$. It is easy to show that the corresponding cdf is

$$F(x) = \begin{cases} 0.5e^{x}, & \text{if } x < 0, \\ 1 - 0.5e^{-x}, & \text{if } x \geq 0. \end{cases}$$

Take $g(t) = 1$. Then

$$\lim_{t\to\infty} \frac{1 - F\left(t + xg(t)\right)}{1 - F(t)} = \lim_{t\to\infty} \frac{e^{-t-x}}{e^{-t}} = e^{-x}.$$

So, $f = 0.5e^{-|x|}$ belongs to the Gumbel domain of attraction.

(4 marks)

UNSEEN

e) Note that $w(F) = \infty$. Then

$$\lim_{t\uparrow\infty} \frac{1 - F\left(tx\right)}{1 - F(t)} = \lim_{t\uparrow\infty} \frac{1 - \exp\left(-(tx)^{-1}\right)}{1 - \exp\left(-t^{-1}\right)} = \lim_{t\uparrow\infty} \frac{1 - \left(1 - (tx)^{-1}\right)}{1 - \left(1 - t^{-1}\right)} = \lim_{t\uparrow\infty} \frac{(tx)^{-1}}{t^{-1}} = x^{-1}.$$

. So, the cdf $F(x) = \exp\left(-x^{-1}\right)$ belongs to the Fréchet domain of attraction.

(4 marks)

UNSEEN

Solutions to Question B4

(a) If X is an absolutely continuous random variable with cdf $F(\cdot)$ then

$$\mathrm{VaR}_p(X) = F^{-1}(p)$$

and

$$\mathrm{ES}_p(X) = \frac{1}{p}\int_0^p F^{-1}(v)dv.$$

(2 marks)

SEEN

(b) (i) The corresponding cdf is

$$F(x) = \int_K^x aK^a y^{-a-1} dy = \left[-K^a y^{-a}\right]_K^x = 1 - K^a x^{-a}$$

for $x > K$;

(2 marks)

SEEN

(b) (ii) Inverting

$$F(x) = 1 - K^a x^{-a} = p,$$

we obtain $\mathrm{VaR}_p(X) = K(1-p)^{-1/a}$.

(2 marks)

SEEN

(b) (iii) The expected shortfall is

$$\mathrm{ES}_p(X) = \frac{K}{p}\int_0^p (1-v)^{-1/a} dv = \frac{K}{p}\left[\frac{(1-v)^{1-1/a}}{\frac{1}{a}-1}\right]_0^p = \frac{aK}{p(1-a)}\left[(1-p)^{1-\frac{1}{a}} - 1\right].$$

(2 marks)

SEEN

c) (i) The joint likelihood function of a and K is

$$L(a, K) = \prod_{i=1}^n \left[aK^a x_i^{-a-1} I\left\{x_i \geq K\right\}\right] = a^n K^{na} \left(\prod_{i=1}^n x_i\right)^{-a-1} I\left\{\min x_i \geq K\right\}.$$

(1 marks)

UNSEEN

c) (ii) Note that $L(a, K)$ is an increasing function over $K \in [0, \min x_i]$ and is zero elsewhere. So, the mle of K is $\widehat{K} = \min x_i$.

(2 marks)

UNSEEN

c) (iii) The log-likelihood function corresponding to $L(a, K)$ is

$$\log L(a, K) = n \log a + na \log K - (a+1) \sum_{i=1}^{n} \log x_i + \log I \left\{ \min x_i \geq K \right\}.$$

So,

$$\frac{d \log L}{da} = \frac{n}{a} + n \log K - \sum_{i=1}^{n} \log x_i.$$

Setting this to zero gives

$$\widehat{a} = n \left[\sum_{i=1}^{n} \log x_i - n \log \min x_i \right]^{-1}.$$

(2 marks)

UNSEEN

c) (iv) The mle of VaR is $\widehat{\mathrm{VaR}}_0(X) = \widehat{K} = \min x_i$. By L'Hopital's rule, the MLE of ES is also $\widehat{\mathrm{ES}}_0(X) = \widehat{K} = \min x_i$.

(2 marks)

UNSEEN

c (v) Let $Z = \min x_i$. The cdf of Z is

$$F_Z(z) = \Pr(Z \leq z) = 1 - \Pr(Z > z) = 1 - \Pr(\min X_i > z) = 1 - \Pr^n(X > z) = 1 - \left(\frac{K}{z} \right)^{na}$$

for $z > K$. The corresponding pdf is

$$f_Z(z) = naK^{na} z^{-na-1}$$

for $z > K$. Z is a Pareto random variable.

(3 marks)

UNSEEN

The expected value of Z is

$$E(Z) = naK^{na} \int_K^\infty z^{-na} dz = naK^{na} \left[\frac{z^{1-na}}{1-na} \right]_K^\infty .$$

So, $\widehat{K}$ and hence $\widehat{\mathrm{VaR}}_0(X)$, $\widehat{\mathrm{ES}}_0(X)$ are biased.

(2 marks)

UNSEEN

Solutions to Question B5

a) The cdf of Y is

$$
\begin{aligned}
F_Y(y) &= \Pr(Y \leq y) \\
&= \Pr\left(\max\left(X_1, \ldots, X_m\right) \leq y\right) \\
&= \Pr\left(X_1 \leq y, \ldots, X_m \leq y\right) \\
&= \Pr\left(X_1 \leq y\right) \cdots \Pr\left(X_m \leq y\right) \\
&= \left(\frac{y-a}{b-a}\right) \cdots \left(\frac{y-a}{b-a}\right) \\
&= \left(\frac{y-a}{b-a}\right)^m
\end{aligned}
$$

for $y > 0$.

(2 marks)

UNSEEN

b) The corresponding pdf is

$$
f_Y(y) = \frac{m}{b-a} \left(\frac{y-a}{b-a}\right)^{m-1}.
$$

for $y > 0$.

(2 marks)

UNSEEN

c) The nth moment of Y can be calculated as

$$
\begin{aligned}
E\left(Y^n\right) &= \int_a^b y^n \frac{m}{b-a} \left(\frac{y-a}{b-a}\right)^{m-1} dy \\
&= m(b-a)^{-m} \int_a^b (y-a+a)^n (y-a)^{m-1} dy \\
&= m(b-a)^{-m} \int_a^b \sum_{k=0}^{n} \binom{n}{k} a^{n-k} (y-a)^{k+m-1} dy \\
&= m(b-a)^{-m} \sum_{k=0}^{n} \binom{n}{k} a^{n-k} \int_a^b (y-a)^{k+m-1} dy \\
&= m(b-a)^{-m} \sum_{k=0}^{n} \binom{n}{k} a^{n-k} \left[\frac{(y-a)^{k+m}}{k+m}\right]_a^b \\
&= m(b-a)^{-m} \sum_{k=0}^{n} \binom{n}{k} a^{n-k} \frac{(b-a)^{k+m}}{k+m}.
\end{aligned}
$$

So,

$$E\left(Y\right) = m(b-a)^{-m} \sum_{k=0}^{1} a^{1-k} \frac{(b-a)^{k+m}}{k+m}$$

and

$$Var\left(Y\right) = m(b-a)^{-m} \sum_{k=0}^{2} \binom{2}{k} a^{2-k} \frac{(b-a)^{k+m}}{k+m} - E^2(Y).$$

(4 marks)

UNSEEN

d) Setting

$$\left(\frac{y-a}{b-a}\right)^{m} = p$$

gives

$$\mathrm{VaR}_p(Y) = a + (b-a)p^{1/m}.$$

(2 marks)

UNSEEN

e) The expected shortfall is

$$\begin{aligned}
\mathrm{ES}_p(Y) &= \frac{1}{p}\int_0^p \left(a + (b-a)v^{1/m}\right) dv \\
&= \frac{1}{p}\left[av + (b-a)\frac{v^{1+1/m}}{1+1/m}\right]_0^p \\
&= \frac{1}{p}\left[ap + (b-a)\frac{p^{1+1/m}}{1+1/m}\right] \\
&= a + (b-a)\frac{p^{1/m}}{1+1/m}.
\end{aligned}$$

(4 marks)

UNSEEN

f) The likelihood and log likelihood functions are

$$\begin{aligned}
L(a,b) &= \prod_{i=1}^{n}\left[\frac{m}{b-a}\left(\frac{y_i-a}{b-a}\right)^{m-1} I\left\{a<y_i<b\right\}\right] \\
&= m^n(b-a)^{-mn}\left[\prod_{i=1}^{n}(y_i-a)\right]^{m-1}\prod_{i=1}^{n} I\left\{a<y_i<b\right\} \\
&= m^n(b-a)^{-mn}\left[\prod_{i=1}^{n}(y_i-a)\right]^{m-1} I\left\{\min y_i>a, \max y_i<b\right\}
\end{aligned}$$

and

$$\log L(a,b) = n\log m - mn\log(b-a) + (m-1)\sum_{i=1}^{n}\log\left(y_i-a\right) + \sum_{i=1}^{n}\log I\left\{\min y_i>a, \max y_i<b\right\}.$$

The likelihood function is a decreasing function of b, so it MLE is $\max y_i$. The partial derivative of the log-likelihood with respect to a is

$$\frac{\partial \log L(a,b)}{\partial a} = \frac{mn}{b-a} - (m-1)\sum_{i=1}^{n}\frac{1}{y_i-a}.$$

So, the MLE of a is the root of

$$\frac{mn}{b-a} = (m-1)\sum_{i=1}^{n}\frac{1}{y_i-a}.$$

(6 marks)

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