

**SOLUTIONS TO
MATH48181
EXTREME VALUES
AND FINANCIAL RISK EXAM**

Solutions to Question A1

a) The marginal cdfs of

$$F_{X,Y}(x, y) = [1 + \exp(-x) + \exp(-y) + (1 - \alpha) \exp(-x - y)]^{-1}$$

are

$$F_X(x) = F_{X,Y}(x, \infty) = [1 + \exp(-x)]^{-1}$$

and

$$F_Y(y) = F_{X,Y}(\infty, y) = [1 + \exp(-y)]^{-1}.$$

(3 marks)

UNSEEN

b) F_X belongs to the Gumbel max domain of attraction since

$$\begin{aligned} \lim_{t \rightarrow w(F_X)} \frac{1 - F_X(t+x)}{1 - F_X(t)} &= \lim_{t \rightarrow \infty} \frac{1 - [1 + e^{-t-x}]^{-1}}{1 - [1 + e^{-t}]^{-1}} \\ &= \lim_{t \rightarrow \infty} \frac{1 - [1 - e^{-t-x}]}{1 - [1 - e^{-t}]} \\ &= e^{-x}. \end{aligned}$$

(3 marks)

c) F_Y belongs to the Gumbel max domain of attraction since

$$\begin{aligned} \lim_{t \rightarrow w(F_Y)} \frac{1 - F_Y(t+x)}{1 - F_Y(t)} &= \lim_{t \rightarrow \infty} \frac{1 - [1 + e^{-t-y}]^{-1}}{1 - [1 + e^{-t}]^{-1}} \\ &= \lim_{t \rightarrow \infty} \frac{1 - [1 - e^{-t-y}]}{1 - [1 - e^{-t}]} \\ &= e^{-y}. \end{aligned}$$

(2 marks)

SEEN

d) Use the rule $a_n = \gamma^{-1} (F_X^{-1}(1 - n^{-1}))$ and $b_n = F_X^{-1}(1 - n^{-1})$. From part (b), $\gamma(t) = 1$. Inverting

$$F_X(x) = [1 + \exp(-x)]^{-1} = 1 - n^{-1},$$

we obtain

$$F_X^{-1}(1 - n^{-1}) = \log(n - 1).$$

(3 marks)

UNSEEN

e) Use the rule $c_n = \gamma^{-1}(F_Y^{-1}(1 - n^{-1}))$ and $d_n = F_Y^{-1}(1 - n^{-1})$. From part (c), $\gamma(t) = 1$. Inverting

$$F_Y(y) = [1 + \exp(-y)]^{-1} = 1 - n^{-1},$$

we obtain

$$F_Y^{-1}(1 - n^{-1}) = \log(n - 1).$$

(2 marks)

UNSEEN

g) The limiting cdf is

$$\begin{aligned} & \lim_{n \rightarrow \infty} F_{X,Y}(a_n x + b_n, c_n y + d_n) \\ &= \lim_{n \rightarrow \infty} [1 + \exp(-x - \log(n - 1)) + \exp(-y - \log(n - 1)) + (1 - \alpha) \exp(-x - y - 2 \log(n - 1))]^{-n} \\ &= \lim_{n \rightarrow \infty} \left[1 + \frac{\exp(-x)}{n - 1} + \frac{\exp(-y)}{n - 1} + (1 - \alpha) \frac{\exp(-x - y)}{(n - 1)^2} \right]^{-n} \\ &= \exp\{-\exp(-x) - \exp(-y)\}. \end{aligned}$$

(5 marks)

UNSEEN

h) Yes, the extremes are completely independent since

$$\begin{aligned} \lim_{n \rightarrow \infty} F_{X,Y}(a_n x + b_n, c_n y + d_n) &= \exp\{-\exp(-x) - \exp(-y)\} \\ &= \lim_{n \rightarrow \infty} F_X(a_n x + b_n) \lim_{n \rightarrow \infty} F_Y(c_n y + d_n). \end{aligned}$$

(2 marks)

UNSEEN

Solutions to Question A2

$C(u_1, u_2)$ is a valid copula if

$$C(u, 0) = 0,$$

$$C(0, u) = 0,$$

$$C(1, u) = u,$$

$$C(u, 1) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) \geq 0.$$

(4 marks)

SEEN

a) for the copula defined by $C(u_1, u_2) = [\alpha(\min(u_1, u_2))^m + (1 - \alpha)u_1^m u_2^m]^{1/m}$, we have

$$C(u, 0) = [\alpha \cdot 0 + (1 - \alpha) \cdot 0]^{1/m} = 0,$$

$$C(0, u) = [\alpha \cdot 0 + (1 - \alpha) \cdot 0]^{1/m} = 0,$$

$$C(1, u) = [\alpha u^m + (1 - \alpha)u^m]^{1/m} = u,$$

$$C(u, 1) = [\alpha u^m + (1 - \alpha)u^m]^{1/m} = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \begin{cases} [\alpha + (1 - \alpha)u_2^m]^{1/m}, & \text{if } u_1 \leq u_2, \\ (1 - \alpha)u_1^{m-1}u_2[\alpha + (1 - \alpha)u_2^m]^{1/m-1}, & \text{if } u_2 \leq u_1 \end{cases} \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \begin{cases} (1 - \alpha)u_1u_2^{m-1}[\alpha + (1 - \alpha)u_2^m]^{1/m-1}, & \text{if } u_1 \leq u_2, \\ [\alpha + (1 - \alpha)u_2^m]^{1/m}, & \text{if } u_2 \leq u_1 \end{cases} \geq 0.$$

(4 marks)

b) for the copula defined by $C(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$, we have

$$C(u, 0) = \max(u + 0 - 1, 0) = 0,$$

$$C(0, u) = \max(0 + u - 1, 0) = 0,$$

$$C(1, u) = \max(1 + u - 1, 0) = u,$$

$$C(u, 1) = \max(u + 1 - 1, 0) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \begin{cases} 1, & \text{if } u_1 + u_2 \geq 1, \\ 0, & \text{if } u_1 + u_2 < 1 \end{cases} \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \begin{cases} 1, & \text{if } u_1 + u_2 \geq 1, \\ 0, & \text{if } u_1 + u_2 < 1 \end{cases} \geq 0.$$

(4 marks)

c) for the copula defined by $C(u_1, u_2) = \min(u_1^a, u_2^b) \min(u_1^{1-a}, u_2^{1-b})$, we have

$$C(u, 0) = \min(u^a, 0) \min(u^{1-a}, 0) = 0,$$

$$C(0, u) = \min(0, u^b) \min(0, u^{1-b}) = 0,$$

$$C(1, u) = \min(1, u^b) \min(1, u^{1-b}) = u,$$

$$C(u, 1) = \min(u^a, 1) \min(u^{1-a}, 1) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \begin{cases} 1, & \text{if } u_1^a \leq u_2^b \text{ and } u_1^{1-a} \leq u_2^{1-b}, \\ au_1^{a-1}u_2^{1-b}, & \text{if } u_1^a \leq u_2^b \text{ and } u_1^{1-a} > u_2^{1-b}, \\ (1-a)u_1^{-a}u_2^b, & \text{if } u_1^a > u_2^b \text{ and } u_1^{1-a} \leq u_2^{1-b}, \\ 0, & \text{if } u_1^a > u_2^b \text{ and } u_1^{1-a} > u_2^{1-b} \end{cases} \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \begin{cases} 0, & \text{if } u_1^a \leq u_2^b \text{ and } u_1^{1-a} \leq u_2^{1-b}, \\ (1-b)u_1^a u_2^{-b}, & \text{if } u_1^a \leq u_2^b \text{ and } u_1^{1-a} > u_2^{1-b}, \\ bu_1^{1-a} u_2^{b-1}, & \text{if } u_1^a > u_2^b \text{ and } u_1^{1-a} \leq u_2^{1-b}, \\ 1, & \text{if } u_1^a > u_2^b \text{ and } u_1^{1-a} > u_2^{1-b} \end{cases} \geq 0.$$

(4 marks)

d) for the copula defined by $C(u_1, u_2) = \exp \left\{ - \left[(-\log u_1)^\theta + (-\log u_2)^\theta \right]^{1/\theta} \right\}$, we have

$$C(u, 0) = \exp \left\{ - \left[(-\log u)^\theta + (\infty)^\theta \right]^{1/\theta} \right\} = 0,$$

$$C(0, u) = \exp \left\{ - \left[(\infty)^\theta + (-\log u)^\theta \right]^{1/\theta} \right\} = 0,$$

$$C(1, u) = \exp \left\{ - \left[0 + (-\log u)^\theta \right]^{1/\theta} \right\} = u,$$

$$C(u, 1) = \exp \left\{ - \left[(-\log u)^\theta + 0 \right]^{1/\theta} \right\} = u,$$

$$\begin{aligned} \frac{\partial}{\partial u_1} C(u_1, u_2) &= u_1^{-1} (-\log u_1)^{\theta-1} \left[(-\log u_1)^\theta + (-\log u_2)^\theta \right]^{1/\theta-1} \\ &\cdot \exp \left\{ - \left[(-\log u_1)^\theta + (-\log u_2)^\theta \right]^{1/\theta} \right\} \geq 0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial u_2} C(u_1, u_2) &= u_2^{-1} (-\log u_2)^{\theta-1} \left[(-\log u_1)^\theta + (-\log u_2)^\theta \right]^{1/\theta-1} \\ &\cdot \exp \left\{ - \left[(-\log u_1)^\theta + (-\log u_2)^\theta \right]^{1/\theta} \right\} \geq 0. \end{aligned}$$

(4 marks)

Solutions to Question A3

a) We can write

$$\bar{F}(x, y) = \exp \left\{ -(x + y) \left[1 - (\theta + \phi) \frac{y}{x + y} + \frac{\theta y^2}{(x + y)^2} + \frac{\phi y^3}{(x + y)^3} \right] \right\}$$

for $x > 0$, $y > 0$, $\theta \geq 0$, $\phi \geq 0$, $\theta + 3\phi \geq 0$, $\theta + \phi \leq 1$ and $\theta + 2\phi \leq 1$. This is in the form of

$$\bar{F}(x, y) = \exp \left[-(x + y) A \left(\frac{y}{x + y} \right) \right]$$

with $A(w) = 1 - (\theta + \phi)w + \theta w^2 + \phi w^3$.

We now check the conditions for $A(\cdot)$. Clearly, $A(0) = 1$ and $A(1) = 1$.

Also $A(t) \geq 0$ since $\theta + \phi \leq 1$ implies $1 - (\theta + \phi)w \leq 1$ for all w .

Also $A(w) \leq 1$ since

$$\begin{aligned} A(w) &\leq 1 \\ \Leftrightarrow 1 - (\theta + \phi)w + \theta w^2 + \phi w^3 &\leq 1 \\ \Leftrightarrow \theta(w^2 - w) + \phi(w^3 - w) &\leq 0 \\ \Leftrightarrow \theta w(w - 1) + \phi w(w^2 - 1) &\leq 0 \\ \Leftrightarrow (\theta + \phi + \phi w)w(w - 1) &\leq 0. \end{aligned}$$

Note that $\max(w, 1 - w) = w$ if $w \in [1/2, 1]$. So, for $w \in [1/2, 1]$,

$$\begin{aligned} A(w) &\geq \max(w, 1 - w) \\ \Leftrightarrow A(w) &\geq w \\ \Leftrightarrow 1 - (\theta + \phi + 1)w + \theta w^2 + \phi w^3 &\geq 0. \end{aligned}$$

Let $g(w) = 1 - (\theta + \phi + 1)w + \theta w^2 + \phi w^3$. Note that $g'(w) = 2\theta w + 3\phi w^2 - \theta - \phi - 1 \leq 0$ for all $w \in [1/2, 1]$. But $g(1) = 0 \geq 0$, so $A(w) \geq \max(w, 1 - w)$ for all $w \in [1/2, 1]$.

Note that $\max(w, 1 - w) = 1 - w$ if $w \in [0, 1/2]$. So, for $w \in [0, 1/2]$,

$$\begin{aligned} A(w) &\geq \max(w, 1 - w) \\ \Leftrightarrow A(w) &\geq 1 - w \\ \Leftrightarrow (1 - \theta - \phi)w + \theta w^2 + \phi w^3 &\geq 0, \end{aligned}$$

which holds since $\theta + \phi \leq 1$.

$A(\cdot)$ is convex since

$$A'(w) = 2\theta w + 3\phi w^2 - \theta - \phi$$

and

$$A''(t) = 2\theta + 6\phi w \geq 0$$

for all w since $\theta + 3\phi \geq 0$.

(6 marks)

UNSEEN

b) the joint cdf is

$$F(x, y) = 1 - \exp(-x) - \exp(-y) + \exp\left\{-(x+y) \left[1 - (\theta + \phi)\frac{y}{x+y} + \frac{\theta y^2}{(x+y)^2} + \frac{\phi y^3}{(x+y)^3}\right]\right\}.$$

(2 marks)

UNSEEN

c) the derivative of joint cdf with respect to x is

$$\frac{\partial F(x, y)}{\partial x} = \exp(-x) + \bar{F}(x, y) \left[\frac{\theta y^2}{(x+y)^2} + \frac{2\phi y^3}{(x+y)^3} - 1 \right],$$

so the conditional cdf if Y given $X = x$ is

$$F(y|x) = 1 + \exp(x)\bar{F}(x, y) \left[\frac{\theta y^2}{(x+y)^2} + \frac{2\phi y^3}{(x+y)^3} - 1 \right].$$

(4 marks)

UNSEEN

d) the derivative of joint cdf with respect to y is

$$\frac{\partial F(x, y)}{\partial y} = \exp(-y) + \bar{F}(x, y) \left[\frac{2\phi y^3}{(x+y)^3} + \frac{(\theta - 3\phi)y^2}{(x+y)^2} - \frac{2\theta y}{x+y} + \theta + \phi - 1 \right],$$

so the conditional cdf if X given $Y = y$ is

$$F(x|y) = 1 + \exp(y)\bar{F}(x, y) \left[\frac{2\phi y^3}{(x+y)^3} + \frac{(\theta - 3\phi)y^2}{(x+y)^2} - \frac{2\theta y}{x+y} + \theta + \phi - 1 \right].$$

(4 marks)

UNSEEN

e) the derivative of joint cdf with respect to x and y is

$$\begin{aligned} f(x, y) &= \frac{\partial F(x, y)}{\partial x \partial y} \\ &= \bar{F}(x, y) \left[\frac{\theta y^2}{(x+y)^2} + \frac{2\phi y^3}{(x+y)^3} - 1 \right] \left[\frac{2\phi y^3}{(x+y)^3} + \frac{(\theta - 3\phi)y^2}{(x+y)^2} - \frac{2\theta y}{x+y} + \theta + \phi - 1 \right] \\ &\quad + \bar{F}(x, y) \left[\frac{2\theta y}{(x+y)^2} + \frac{2(3\phi - \theta)y^2}{(x+y)^3} - \frac{6\theta y^3}{(x+y)^4} \right]. \end{aligned}$$

(4 marks)

UNSEEN

Solutions to Question B1

a) The mgf of X_i is

$$M_{X_i}(t) = \int_0^{\infty} \lambda \exp [-(\lambda - t)x] dx = \left[\frac{\lambda}{t - \lambda} \exp [-(\lambda - t)x] \right]_0^{\infty} = 0 - \frac{\lambda}{t - \lambda} = \frac{\lambda}{\lambda - t}.$$

(2 marks)

SEEN

b) The mgf of T conditional on $N = n$ is

$$\begin{aligned} M_{T|N=n}(t) &= E \{ \exp [t (X_1 + \cdots + X_n)] \} \\ &= E \{ \exp [tX_1] \cdots \exp [tX_n] \} \\ &= E \{ \exp [tX_1] \} \cdots E \{ \exp [tX_n] \} \\ &= M_{X_1}(t) \cdots M_{X_n}(t) \\ &= \left(\frac{\lambda}{\lambda - t} \right)^n \end{aligned}$$

(3 marks)

UNSEEN

c) $\left(\frac{\lambda}{\lambda-t}\right)^n$ is the mgf of a gamma random variable with parameters λ and n . So, the conditional distribution of T is gamma with parameters λ and n .

(3 marks)

UNSEEN

d) The conditional pdf of T is

$$f_{T|N=n}(x) = \frac{\lambda^n x^{n-1} \exp(-\lambda x)}{\Gamma(n)}.$$

So, the unconditional pdf of T is

$$\begin{aligned}f_T(x) &= \sum_{n=1}^{\infty} f_{T|N=n}(x)\theta(1-\theta)^{n-1} \\&= \sum_{n=1}^{\infty} \frac{\lambda^n x^{n-1} \exp(-\lambda x)}{\Gamma(n)} \theta(1-\theta)^{n-1} \\&= \theta \lambda \exp(-\lambda x) \sum_{n=1}^{\infty} \frac{(\lambda(1-\theta)x)^{n-1}}{(n-1)!} \\&= \theta \lambda \exp(-\lambda x) \exp[\lambda x(1-\theta)] \\&= \theta \lambda \exp(-\theta \lambda x),\end{aligned}$$

an Exponential pdf with parameter $\theta\lambda$.

(3 marks)

UNSEEN

e) The mean is $1/(\theta\lambda)$ and the variance $1/(\theta\lambda)^2$.

(3 marks)

UNSEEN

f) The unconditional cdf of T is

$$F_T(x) = 1 - \exp(-\theta\lambda x).$$

Inverting

$$1 - \exp(-\theta\lambda x) = p,$$

we obtain

$$\text{VaR}_p(T) = -\frac{1}{\theta\lambda} \log(1-p).$$

(3 marks)

UNSEEN

g) The expected shortfall T is

$$\begin{aligned} \text{ES}_p(T) &= -\frac{1}{\theta\lambda p} \int_0^p \log(1-t) dt \\ &= -\frac{1}{\theta\lambda p} \left\{ [t \log(1-t)]_0^p + \int_0^p \frac{t}{1-t} dt \right\} \\ &= -\frac{1}{\theta\lambda p} \left\{ p \log(1-p) + \int_0^p \frac{t-1+1}{1-t} dt \right\} \\ &= -\frac{1}{\theta\lambda p} \left\{ p \log(1-p) - p + \int_0^p \frac{1}{1-t} dt \right\} \\ &= -\frac{1}{\theta\lambda p} \{ p \log(1-p) - p + [-\log(1-t)]_0^p \} \\ &= -\frac{1}{\theta\lambda p} \{ p \log(1-p) - p - \log(1-p) \}. \end{aligned}$$

(3 marks)

UNSEEN

Solutions to Question B2

If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cdf of a normalized version of M_n converges to G , i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n(a_n x + b_n) \rightarrow G(x) \quad (1)$$

as $n \rightarrow \infty$ then G must be of the same type as (cdf's G and G^* are of the same type if $G^*(x) = G(ax + b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$\begin{aligned} I & : \Lambda(x) = \exp\{-\exp(-x)\}, \quad x \in \mathfrak{R}; \\ II & : \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\{-x^{-\alpha}\} & \text{if } x \geq 0 \end{cases} \\ & \quad \text{for some } \alpha > 0; \\ III & : \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^\alpha\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \\ & \quad \text{for some } \alpha > 0. \end{aligned}$$

(6 marks)

SEEN

The necessary and sufficient conditions for the three extreme value distributions are:

$$\begin{aligned} I & : \exists \gamma(t) > 0 \text{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = \exp(-x), \quad x \in \mathfrak{R}, \\ II & : w(F) = \infty \text{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \quad x > 0, \\ III & : w(F) < \infty \text{ and } \lim_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} = x^\alpha, \quad x > 0. \end{aligned}$$

(6 marks)

SEEN

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function say $h(t)$ such that

$$\lim_{t \rightarrow w(G)} \frac{1 - G(t + xh(t))}{1 - G(t)} = e^{-x}$$

for every $x \in (-\infty, \infty)$. But

$$\begin{aligned} \lim_{t \rightarrow W(F)} \frac{1 - F(t + xh(t))}{1 - F(t)} &= \lim_{t \rightarrow w(F)} \left\{ \frac{1 - \{1 - [1 - G(t + xh(t))]^a\}^b}{1 - \{1 - [1 - G(t)]^a\}^b} \right\}^\theta \\ &= \lim_{t \rightarrow w(F)} \left\{ \frac{1 - G(t + xh(t))}{1 - G(t)} \right\}^{a\theta} \\ &= e^{-a\theta x} \end{aligned}$$

for every $x \in (-\infty, \infty)$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n \rightarrow \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp[-\exp(-a\theta x)]$$

for some suitable norming constants $a_n > 0$ and b_n .

(3 marks)

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t \rightarrow \infty} \frac{1 - G(tx)}{1 - G(t)} = x^{-\beta}$$

for every $x > 0$. But

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1 - F(tx)}{1 - F(t)} &= \lim_{t \rightarrow \infty} \left\{ \frac{1 - \{1 - [1 - G(tx)]^a\}^b}{1 - \{1 - [1 - G(t)]^a\}^b} \right\}^\theta \\ &= \lim_{t \rightarrow \infty} \left\{ \frac{1 - G(tx)}{1 - G(t)} \right\}^{a\theta} \\ &= x^{\beta a\theta} \end{aligned}$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n \rightarrow \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp(-x^{-\beta a\theta})$$

for some suitable norming constants $a_n > 0$ and b_n .

(3 marks)

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t \rightarrow 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} = x^\beta$$

for every $x > 0$. But But

$$\begin{aligned}
 \lim_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} &= \lim_{t \downarrow 0} \left\{ \frac{1 - \{1 - [1 - G(w(F) - tx)]^a\}^b}{1 - \{1 - [1 - G(w(F) - t)]^a\}^b} \right\}^\theta \\
 &= \lim_{t \rightarrow \infty} \left\{ \frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \right\}^{a\theta} \\
 &= x^{\beta a \theta}
 \end{aligned}$$

for every $x < 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{n \rightarrow \infty} P \left(\frac{M_n - b_n}{a_n} \leq x \right) = \exp(-(-x)^{\beta a \theta})$$

for some suitable norming constants $a_n > 0$ and b_n .

(2 marks)

UNSEEN

Solutions to Question B3

a) Note that $w(F) = 1$. Then

$$\begin{aligned} \lim_{t \downarrow 0} \frac{1 - F(1 - tx)}{1 - F(1 - t)} &= \lim_{t \downarrow 0} \frac{xf(1 - tx)}{f(1 - t)} \\ &= \lim_{t \downarrow 0} \frac{x(1 - tx)^{\alpha-1} (tx)^{\beta-1}}{(1 - t)^{\alpha-1} t^{\beta-1}} \\ &= x^\beta. \end{aligned}$$

So, $f(x) = Cx^{\alpha-1}(1 - x)^{\beta-1}$ belongs to the Weibull domain of attraction.

(4 marks)

UNSEEN

b) Note that

$$F(k) = \begin{cases} 1/2, & \text{if } k = -1, 0, \\ 1, & \text{if } k \geq 1, \\ 0, & \text{otherwise} \end{cases}$$

and

$$1 - F(k - 1) = \begin{cases} 1, & \text{if } k \leq 0, \\ 1/2, & \text{if } k = 1, 2, \\ 0, & \text{otherwise.} \end{cases}$$

So,

$$\frac{p(k)}{1 - F(k - 1)} = \begin{cases} 1/2, & \text{if } k = -1, \\ 1, & \text{if } k = 1, \\ 0, & \text{if } k \leq 2, k \neq -1, 1, \\ \text{undefined,} & \text{otherwise.} \end{cases}$$

Hence, there can be no non-degenerate limit.

(4 marks)

UNSEEN

c) Note that $w(F) = \infty$. Then

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1 - F(tx)}{1 - F(t)} &= \lim_{t \rightarrow \infty} \frac{xf(tx)}{f(t)} \\ &= \lim_{t \rightarrow \infty} \frac{x(1 + t^2x^2)^{-1}}{(1 + t^2)^{-1}} \\ &= x^{-1}. \end{aligned}$$

So, $f(x) = \pi^{-1}(1+x^2)^{-1}$ belongs to the Fréchet domain of attraction.

(4 marks)

UNSEEN

d) Note that $w(F) = \infty$. It is easy to show that the corresponding cdf is

$$F(x) = \begin{cases} 0.5e^x, & \text{if } x < 0, \\ 1 - 0.5e^{-x}, & \text{if } x \geq 0. \end{cases}$$

Take $g(t) = 1$. Then

$$\lim_{t \rightarrow \infty} \frac{1 - F(t + xg(t))}{1 - F(t)} = \lim_{t \rightarrow \infty} \frac{e^{-t-x}}{e^{-t}} = e^{-x}.$$

So, $f = 0.5e^{-|x|}$ belongs to the Gumbel domain of attraction.

(4 marks)

UNSEEN

e) Note that $w(F) = \infty$. Then

$$\lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = \lim_{t \uparrow \infty} \frac{1 - \exp(-(tx)^{-1})}{1 - \exp(-t^{-1})} = \lim_{t \uparrow \infty} \frac{1 - (1 - (tx)^{-1})}{1 - (1 - t^{-1})} = \lim_{t \uparrow \infty} \frac{(tx)^{-1}}{t^{-1}} = x^{-1}.$$

. So, the cdf $F(x) = \exp(-x^{-1})$ belongs to the Fréchet domain of attraction.

(4 marks)

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Solutions to Question B4

(a) If X is an absolutely continuous random variable with cdf $F(\cdot)$ then

$$\text{VaR}_p(X) = F^{-1}(p)$$

and

$$\text{ES}_p(X) = \frac{1}{p} \int_0^p F^{-1}(v) dv.$$

(2 marks)

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(b) (i) The corresponding cdf is

$$F(x) = \int_K^x aK^a y^{-a-1} dy = [-K^a y^{-a}]_K^x = 1 - K^a x^{-a}$$

for $x > K$;

(2 marks)

SEEN

(b) (ii) Inverting

$$F(x) = 1 - K^a x^{-a} = p,$$

we obtain $\text{VaR}_p(X) = K(1 - p)^{-1/a}$.

(2 marks)

SEEN

(b) (iii) The expected shortfall is

$$\text{ES}_p(X) = \frac{K}{p} \int_0^p (1 - v)^{-1/a} dv = \frac{K}{p} \left[\frac{(1 - v)^{1-1/a}}{\frac{1}{a} - 1} \right]_0^p = \frac{aK}{p(1 - a)} \left[(1 - p)^{1-\frac{1}{a}} - 1 \right].$$

(2 marks)

SEEN

c) (i) The joint likelihood function of a and K is

$$L(a, K) = \prod_{i=1}^n [aK^a x_i^{-a-1} I\{x_i \geq K\}] = a^n K^{na} \left(\prod_{i=1}^n x_i \right)^{-a-1} I\{\min x_i \geq K\}.$$

(1 marks)

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c) (ii) Note that $L(a, K)$ is an increasing function over $K \in [0, \min x_i]$ and is zero elsewhere. So, the mle of K is $\hat{K} = \min x_i$.

(2 marks)

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c) (iii) The log-likelihood function corresponding to $L(a, K)$ is

$$\log L(a, K) = n \log a + na \log K - (a + 1) \sum_{i=1}^n \log x_i + \log I \{ \min x_i \geq K \}.$$

So,

$$\frac{d \log L}{da} = \frac{n}{a} + n \log K - \sum_{i=1}^n \log x_i.$$

Setting this to zero gives

$$\hat{a} = n \left[\sum_{i=1}^n \log x_i - n \log \min x_i \right]^{-1}.$$

(2 marks)

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c) (iv) The mle of VaR is $\widehat{\text{VaR}}_0(X) = \hat{K} = \min x_i$. By L'Hopital's rule, the MLE of ES is also $\widehat{\text{ES}}_0(X) = \hat{K} = \min x_i$.

(2 marks)

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c) (v) Let $Z = \min x_i$. The cdf of Z is

$$F_Z(z) = \Pr(Z \leq z) = 1 - \Pr(Z > z) = 1 - \Pr(\min X_i > z) = 1 - \Pr^n(X > z) = 1 - \left(\frac{K}{z}\right)^{na}$$

for $z > K$. The corresponding pdf is

$$f_Z(z) = naK^{na}z^{-na-1}$$

for $z > K$. Z is a Pareto random variable.

(3 marks)

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The expected value of Z is

$$E(Z) = naK^{na} \int_K^{\infty} z^{-na} dz = naK^{na} \left[\frac{z^{1-na}}{1-na} \right]_K^{\infty}.$$

So, \widehat{K} and hence $\widehat{\text{VaR}}_0(X)$, $\widehat{\text{ES}}_0(X)$ are biased.

(2 marks)

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Solutions to Question B5

a) The cdf of Y is

$$\begin{aligned}F_Y(y) &= \Pr(Y \leq y) \\&= \Pr(\max(X_1, \dots, X_m) \leq y) \\&= \Pr(X_1 \leq y, \dots, X_m \leq y) \\&= \Pr(X_1 \leq y) \cdots \Pr(X_m \leq y) \\&= \left(\frac{y-a}{b-a}\right) \cdots \left(\frac{y-a}{b-a}\right) \\&= \left(\frac{y-a}{b-a}\right)^m\end{aligned}$$

for $y > 0$.

(2 marks)

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b) The corresponding pdf is

$$f_Y(y) = \frac{m}{b-a} \left(\frac{y-a}{b-a}\right)^{m-1}.$$

for $y > 0$.

(2 marks)

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c) The n th moment of Y can be calculated as

$$\begin{aligned}E(Y^n) &= \int_a^b y^n \frac{m}{b-a} \left(\frac{y-a}{b-a}\right)^{m-1} dy \\&= m(b-a)^{-m} \int_a^b (y-a+a)^n (y-a)^{m-1} dy \\&= m(b-a)^{-m} \int_a^b \sum_{k=0}^n \binom{n}{k} a^{n-k} (y-a)^{k+m-1} dy \\&= m(b-a)^{-m} \sum_{k=0}^n \binom{n}{k} a^{n-k} \int_a^b (y-a)^{k+m-1} dy \\&= m(b-a)^{-m} \sum_{k=0}^n \binom{n}{k} a^{n-k} \left[\frac{(y-a)^{k+m}}{k+m} \right]_a^b \\&= m(b-a)^{-m} \sum_{k=0}^n \binom{n}{k} a^{n-k} \frac{(b-a)^{k+m}}{k+m}.\end{aligned}$$

So,

$$E(Y) = m(b-a)^{-m} \sum_{k=0}^1 a^{1-k} \frac{(b-a)^{k+m}}{k+m}$$

and

$$Var(Y) = m(b-a)^{-m} \sum_{k=0}^2 \binom{2}{k} a^{2-k} \frac{(b-a)^{k+m}}{k+m} - E^2(Y).$$

(4 marks)

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d) Setting

$$\left(\frac{y-a}{b-a}\right)^m = p$$

gives

$$VaR_p(Y) = a + (b-a)p^{1/m}.$$

(2 marks)

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e) The expected shortfall is

$$\begin{aligned} ES_p(Y) &= \frac{1}{p} \int_0^p (a + (b-a)v^{1/m}) dv \\ &= \frac{1}{p} \left[av + (b-a) \frac{v^{1+1/m}}{1+1/m} \right]_0^p \\ &= \frac{1}{p} \left[ap + (b-a) \frac{p^{1+1/m}}{1+1/m} \right] \\ &= a + (b-a) \frac{p^{1/m}}{1+1/m}. \end{aligned}$$

(4 marks)

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f) The likelihood and log likelihood functions are

$$\begin{aligned}
 L(a, b) &= \prod_{i=1}^n \left[\frac{m}{b-a} \left(\frac{y_i - a}{b-a} \right)^{m-1} I \{a < y_i < b\} \right] \\
 &= m^n (b-a)^{-mn} \left[\prod_{i=1}^n (y_i - a) \right]^{m-1} \prod_{i=1}^n I \{a < y_i < b\} \\
 &= m^n (b-a)^{-mn} \left[\prod_{i=1}^n (y_i - a) \right]^{m-1} I \{\min y_i > a, \max y_i < b\}
 \end{aligned}$$

and

$$\log L(a, b) = n \log m - mn \log(b-a) + (m-1) \sum_{i=1}^n \log(y_i - a) + \sum_{i=1}^n \log I \{\min y_i > a, \max y_i < b\}.$$

The likelihood function is a decreasing function of b , so its MLE is $\max y_i$. The partial derivative of the log-likelihood with respect to a is

$$\frac{\partial \log L(a, b)}{\partial a} = \frac{mn}{b-a} - (m-1) \sum_{i=1}^n \frac{1}{y_i - a}.$$

So, the MLE of a is the root of

$$\frac{mn}{b-a} = (m-1) \sum_{i=1}^n \frac{1}{y_i - a}.$$

(6 marks)

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