SOLUTIONS TO MATH38181 EXTREME VALUES AND FINANCIAL RISK EXAM

Solutions to Question 1

a) The mgf of X_i is

$$M_{X_i}(t) = \int_0^\infty \lambda \exp\left[-(\lambda - t)x\right] dx = \left[\frac{\lambda}{t-\lambda} \exp\left[-(\lambda - t)x\right]\right]_0^\infty = 0 - \frac{\lambda}{t-\lambda} = \frac{\lambda}{\lambda - t}.$$

(2 marks)

SEEN

b) The mgf of T conditional on $N = n$ is

$$\begin{aligned} M_{T|N=n}(t) &= E\left\{\exp\left[t\left(X_1 + \cdots + X_n\right)\right]\right\} \\ &= E\left\{\exp\left[tX_1\right] \cdots \exp\left[X_n\right]\right\} \\ &= E\left\{\exp\left[tX_1\right]\right\} \cdots E\left\{\exp\left[X_n\right]\right\} \\ &= M_{X_1}(t) \cdots M_{X_n}(t) \\ &= \left(\frac{\lambda}{\lambda - t}\right)^n \end{aligned}$$

(3 marks)

UNSEEN

c) $\left(\frac{\lambda}{\lambda - t}\right)^n$ is the mgf of a gamma random variable with parameters λ and n. So, the conditional distribution of T is gamma with parameters λ and n.

(3 marks)

UNSEEN

d) The conditional pdf of T is

$$f_{T|N=n}(x) = \frac{\lambda^n x^{n-1} \exp(-\lambda x)}{\Gamma(n)}.$$

So, the unconditional pdf of T is

$$\begin{aligned} f_T(x) &= \sum_{n=1}^{\infty} f_{T|N=n}(x)\theta(1-\theta)^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{\lambda^n x^{n-1}\exp(-\lambda x)}{\Gamma(n)}\theta(1-\theta)^{n-1} \\ &= \theta\lambda\exp(-\lambda x)\sum_{n=1}^{\infty}\frac{(\lambda(1-\theta)x)^{n-1}}{(n-1)!} \\ &= \theta\lambda\exp(-\lambda x)\exp\left[\lambda x(1-\theta)\right] \\ &= \theta\lambda\exp(-\theta\lambda x), \end{aligned}$$

an Exponential pdf with parameter $\theta\lambda$.

(3 marks)

UNSEEN

e) The mean is $1/(\theta\lambda)$ and the variance $1/(\theta\lambda)^2$.

(3 marks)

UNSEEN

f) The unconditional cdf of T is

$$F_T(x) = 1 - \exp(-\theta\lambda x).$$

Inverting

$$1 - \exp(-\theta\lambda x) = p,$$

we obtain

$$\mathrm{VaR}_p(T) = -\frac{1}{\theta\lambda}\log(1-p).$$

(3 marks)

UNSEEN

g) The expected shortfall T is

$$\begin{aligned}
\mathrm{ES}_p(T) &= -\frac{1}{\theta\lambda p}\int_0^p \log(1-t)dt \\
&= -\frac{1}{\theta\lambda p}\left\{[t\log(1-t)]_0^p + \int_0^p \frac{t}{1-t}dt\right\} \\
&= -\frac{1}{\theta\lambda p}\left\{p\log(1-p) + \int_0^p \frac{t-1+1}{1-t}dt\right\} \\
&= -\frac{1}{\theta\lambda p}\left\{p\log(1-p) - p + \int_0^p \frac{1}{1-t}dt\right\} \\
&= -\frac{1}{\theta\lambda p}\left\{p\log(1-p) - p + [-\log(1-t)]_0^p\right\} \\
&= -\frac{1}{\theta\lambda p}\left\{p\log(1-p) - p - \log(1-p)\right\}.
\end{aligned}$$

(3 marks)

UNSEEN

Solutions to Question 2

If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cdf of a normalized version of M_n converges to G, i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n\left(a_n x + b_n\right) \to G(x) \tag{1}$$

as $n \to \infty$ then G must be of the same type as (cdfs G and G^* are of the same type if $G^*(x) = G(ax+b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$\begin{aligned}
I &: \Lambda(x) = \exp\left\{-\exp(-x)\right\}, \qquad x \in \Re; \\
II &: \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\left\{-x^{-\alpha}\right\} & \text{if } x \geq 0 \end{cases} \\
&\quad\ \text{for some } \alpha > 0; \\
III &: \Psi_\alpha(x) = \begin{cases} \exp\left\{-(-x)^\alpha\right\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \\
&\quad\ \text{for some } \alpha > 0.
\end{aligned}$$

(6 marks)

SEEN

The necessary and sufficient conditions for the three extreme value distributions are:

$$\begin{aligned}
I &: \exists \gamma(t) > 0 \text{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F\left(t + x\gamma(t)\right)}{1 - F(t)} = \exp(-x), \qquad x \in \Re, \\
II &: w(F) = \infty \text{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \qquad x > 0, \\
III &: w(F) < \infty \text{ and } \lim_{t \downarrow 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} = x^\alpha, \qquad x > 0.
\end{aligned}$$

(6 marks)

SEEN

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function say $h(t)$ such that

$$\lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} = e^{-x}$$

for every $x \in (-\infty, \infty)$. But

$$
\begin{aligned}
\lim_{t \to W(F)} \frac{1 - F(t + xh(t))}{1 - F(t)} &= \lim_{t \to w(F)} \left\{ \frac{1 - \{1 - [1 - G(t + xh(t))]^a\}^b}{1 - \{1 - [1 - G(t)]^a\}^b} \right\}^\theta \\
&= \lim_{t \to w(F)} \left\{ \frac{1 - G(t + xh(t))}{1 - G(t)} \right\}^{a\theta} \\
&= e^{-a\theta x}
\end{aligned}
$$

for every $x \in (-\infty, \infty)$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$
\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x \right) = \exp\left[-\exp(-a\theta x) \right]
$$

for some suitable norming constants $a_n > 0$ and b_n.

(3 marks)

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$
\lim_{t \to \infty} \frac{1 - G(tx)}{1 - G(t)} = x^{-\beta}
$$

for every $x > 0$. But

$$
\begin{aligned}
\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} &= \lim_{t \to \infty} \left\{ \frac{1 - \{1 - [1 - G(tx)]^a\}^b}{1 - \{1 - [1 - G(t)]^a\}^b} \right\}^\theta \\
&= \lim_{t \to \infty} \left\{ \frac{1 - G(tx)}{1 - G(t)} \right\}^{a\theta} \\
&= x^{\beta a\theta}
\end{aligned}
$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$
\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x \right) = \exp\left(-x^{-\beta a\theta} \right)
$$

for some suitable norming constants $a_n > 0$ and b_n.

(3 marks)

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$
\lim_{t \to 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} = x^\beta
$$

for every $x > 0$. But But

$$
\begin{aligned}
\lim_{t \downarrow 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} &= \lim_{t \downarrow 0} \left\{ \frac{1 - \left\{1 - \left[1 - G\left(w(F) - tx\right)\right]^a\right\}^b}{1 - \left\{1 - \left[1 - G\left(w(F) - t\right)\right]^a\right\}^b} \right\}^\theta \\
&= \lim_{t \to \infty} \left\{ \frac{1 - G\left(w(F) - tx\right)}{1 - G\left(w(F) - t\right)} \right\}^{a\theta} \\
&= x^{\beta a \theta}
\end{aligned}
$$

for every $x < 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$
\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x \right) = \exp\left(-(-x)^{\beta a \theta} \right)
$$

for some suitable norming constants $a_n > 0$ and b_n.

(2 marks)

UNSEEN

Solutions to Question 3

a) Note that $w(F) = 1$. Then

$$\begin{aligned}
\lim_{t \downarrow 0} \frac{1 - F\left(1 - tx\right)}{1 - F(1 - t)} &= \lim_{t \downarrow 0} \frac{x f\left(1 - tx\right)}{f(1 - t)} \\
&= \lim_{t \downarrow 0} \frac{x\left(1 - tx\right)^{\alpha - 1}\left(tx\right)^{\beta - 1}}{(1 - t)^{\alpha - 1} t^{\beta - 1}} \\
&= x^{\beta}.
\end{aligned}$$

So, $f(x) = Cx^{\alpha-1}(1 - x)^{\beta-1}$ belongs to the Weibull domain of attraction.

(4 marks)

UNSEEN

b) Note that

$$F(k) = \begin{cases} 1/2, & \text{if } k = -1, 0, \\ 1, & \text{if } k \geq 1, \\ 0, & \text{otherwise} \end{cases}$$

and

$$1 - F(k - 1) = \begin{cases} 1, & \text{if } k \leq 0, \\ 1/2, & \text{if } k = 1, 2, \\ 0, & \text{otherwise.} \end{cases}$$

So,

$$\frac{p(k)}{1 - F(k - 1)} = \begin{cases} 1/2, & \text{if } k = -1, \\ 1, & \text{if } k = 1, \\ 0, & \text{if } k \leq 2,\ k \neq -1, 1, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Hence, there can be no non-degenerate limit.

(4 marks)

UNSEEN

c) Note that $w(F) = \infty$. Then

$$\begin{aligned}
\lim_{t \to \infty} \frac{1 - F\left(tx\right)}{1 - F(t)} &= \lim_{t \to \infty} \frac{x f\left(tx\right)}{f(t)} \\
&= \lim_{t \to \infty} \frac{x\left(1 + t^2 x^2\right)^{-1}}{\left(1 + t^2\right)^{-1}} \\
&= x^{-1}.
\end{aligned}$$

So, $f(x) = \pi^{-1}\left(1+x^2\right)^{-1}$ belongs to the Fréchet domain of attraction.

(4 marks)

UNSEEN

d) Note that $w(F) = \infty$. It is easy to show that the corresponding cdf is

$$F(x) = \begin{cases} 0.5e^{x}, & \text{if } x < 0, \\ 1 - 0.5e^{-x}, & \text{if } x \geq 0. \end{cases}$$

Take $g(t) = 1$. Then

$$\lim_{t\to\infty} \frac{1 - F\left(t + xg(t)\right)}{1 - F(t)} = \lim_{t\to\infty} \frac{e^{-t-x}}{e^{-t}} = e^{-x}.$$

So, $f = 0.5e^{-|x|}$ belongs to the Gumbel domain of attraction.

(4 marks)

UNSEEN

e) Note that $w(F) = \infty$. Then

$$\lim_{t\uparrow\infty} \frac{1 - F\left(tx\right)}{1 - F(t)} = \lim_{t\uparrow\infty} \frac{1 - \exp\left(-(tx)^{-1}\right)}{1 - \exp\left(-t^{-1}\right)} = \lim_{t\uparrow\infty} \frac{1 - \left(1 - (tx)^{-1}\right)}{1 - \left(1 - t^{-1}\right)} = \lim_{t\uparrow\infty} \frac{(tx)^{-1}}{t^{-1}} = x^{-1}.$$

. So, the cdf $F(x) = \exp\left(-x^{-1}\right)$ belongs to the Fréchet domain of attraction.

(4 marks)

UNSEEN

Solutions to Question 4

(a) If X is an absolutely continuous random variable with cdf $F(\cdot)$ then

$$\mathrm{VaR}_p(X) = F^{-1}(p)$$

and

$$\mathrm{ES}_p(X) = \frac{1}{p}\int_0^p F^{-1}(v)dv.$$

(2 marks)

SEEN

(b) (i) The corresponding cdf is

$$F(x) = \int_K^x aK^a y^{-a-1} dy = \left[-K^a y^{-a}\right]_K^x = 1 - K^a x^{-a}$$

for $x > K$;

(2 marks)

SEEN

(b) (ii) Inverting

$$F(x) = 1 - K^a x^{-a} = p,$$

we obtain $\mathrm{VaR}_p(X) = K(1-p)^{-1/a}$.

(2 marks)

SEEN

(b) (iii) The expected shortfall is

$$\mathrm{ES}_p(X) = \frac{K}{p}\int_0^p (1-v)^{-1/a} dv = \frac{K}{p}\left[\frac{(1-v)^{1-1/a}}{\frac{1}{a}-1}\right]_0^p = \frac{aK}{p(1-a)}\left[(1-p)^{1-\frac{1}{a}} - 1\right].$$

(2 marks)

SEEN

c) (i) The joint likelihood function of a and K is

$$L(a, K) = \prod_{i=1}^n \left[aK^a x_i^{-a-1} I\left\{x_i \geq K\right\}\right] = a^n K^{na}\left(\prod_{i=1}^n x_i\right)^{-a-1} I\left\{\min x_i \geq K\right\}.$$

(1 marks)

UNSEEN

c) (ii) Note that $L(a, K)$ is an increasing function over $K \in [0, \min x_i]$ and is zero elsewhere. So, the mle of K is $\widehat{K} = \min x_i$.

(2 marks)

UNSEEN

c) (iii) The log-likelihood function corresponding to $L(a, K)$ is

$$\log L(a, K) = n \log a + na \log K - (a+1) \sum_{i=1}^{n} \log x_i + \log I\left\{\min x_i \geq K\right\}.$$

So,

$$\frac{d \log L}{da} = \frac{n}{a} + n \log K - \sum_{i=1}^{n} \log x_i.$$

Setting this to zero gives

$$\widehat{a} = n \left[\sum_{i=1}^{n} \log x_i - n \log \min x_i\right]^{-1}.$$

(2 marks)

UNSEEN

c) (iv) The mle of VaR is $\widehat{\mathrm{VaR}}_0(X) = \widehat{K} = \min x_i$. By L'Hopital's rule, the MLE of ES is also $\widehat{\mathrm{ES}}_0(X) = \widehat{K} = \min x_i$.

(2 marks)

UNSEEN

c (v) Let $Z = \min x_i$. The cdf of Z is

$$F_Z(z) = \Pr(Z \leq z) = 1 - \Pr(Z > z) = 1 - \Pr(\min X_i > z) = 1 - \Pr^n(X > z) = 1 - \left(\frac{K}{z}\right)^{na}$$

for $z > K$. The corresponding pdf is

$$f_Z(z) = naK^{na} z^{-na-1}$$

for $z > K$. Z is a Pareto random variable.

(3 marks)

UNSEEN

The expected value of Z is

$$E(Z) = naK^{na} \int_K^\infty z^{-na} dz = naK^{na} \left[\frac{z^{1-na}}{1-na} \right]_K^\infty.$$

So, $\widehat{K}$ and hence $\widehat{\mathrm{VaR}}_0(X)$, $\widehat{\mathrm{ES}}_0(X)$ are biased.

(2 marks)

UNSEEN

Solutions to Question 5

a) The cdf of Y is

$$\begin{aligned}
F_Y(y) &= \Pr(Y \leq y) \\
&= \Pr\left(\max\left(X_1, \ldots, X_m\right) \leq y\right) \\
&= \Pr\left(X_1 \leq y, \ldots, X_m \leq y\right) \\
&= \Pr\left(X_1 \leq y\right) \cdots \Pr\left(X_m \leq y\right) \\
&= \left(\frac{y-a}{b-a}\right) \cdots \left(\frac{y-a}{b-a}\right) \\
&= \left(\frac{y-a}{b-a}\right)^m
\end{aligned}$$

for $y > 0$.

(2 marks)

UNSEEN

b) The corresponding pdf is

$$f_Y(y) = \frac{m}{b-a}\left(\frac{y-a}{b-a}\right)^{m-1}.$$

for $y > 0$.

(2 marks)

UNSEEN

c) The nth moment of Y can be calculated as

$$\begin{aligned}
E\left(Y^n\right) &= \int_a^b y^n \frac{m}{b-a}\left(\frac{y-a}{b-a}\right)^{m-1} dy \\
&= m(b-a)^{-m} \int_a^b (y-a+a)^n (y-a)^{m-1} dy \\
&= m(b-a)^{-m} \int_a^b \sum_{k=0}^n \binom{n}{k} a^{n-k} (y-a)^{k+m-1} dy \\
&= m(b-a)^{-m} \sum_{k=0}^n \binom{n}{k} a^{n-k} \int_a^b (y-a)^{k+m-1} dy \\
&= m(b-a)^{-m} \sum_{k=0}^n \binom{n}{k} a^{n-k} \left[\frac{(y-a)^{k+m}}{k+m}\right]_a^b \\
&= m(b-a)^{-m} \sum_{k=0}^n \binom{n}{k} a^{n-k} \frac{(b-a)^{k+m}}{k+m}.
\end{aligned}$$

So,

$$E\left(Y\right)=m(b-a)^{-m}\sum_{k=0}^{1}a^{1-k}\frac{(b-a)^{k+m}}{k+m}$$

and

$$Var\left(Y\right)=m(b-a)^{-m}\sum_{k=0}^{2}\binom{2}{k}a^{2-k}\frac{(b-a)^{k+m}}{k+m}-E^{2}(Y).$$

(4 marks)

UNSEEN

d) Setting

$$\left(\frac{y-a}{b-a}\right)^{m}=p$$

gives

$$\mathrm{VaR}_p(Y)=a+(b-a)p^{1/m}.$$

(2 marks)

UNSEEN

e) The expected shortfall is

$$\begin{aligned}
\mathrm{ES}_p(Y) &= \frac{1}{p}\int_0^p\left(a+(b-a)v^{1/m}\right)dv \\
&= \frac{1}{p}\left[av+(b-a)\frac{v^{1+1/m}}{1+1/m}\right]_0^p \\
&= \frac{1}{p}\left[ap+(b-a)\frac{p^{1+1/m}}{1+1/m}\right] \\
&= a+(b-a)\frac{p^{1/m}}{1+1/m}.
\end{aligned}$$

(4 marks)

UNSEEN

f) The likelihood and log likelihood functions are

$$
\begin{aligned}
L(a,b) &= \prod_{i=1}^{n}\left[\frac{m}{b-a}\left(\frac{y_i-a}{b-a}\right)^{m-1} I\left\{a<y_i<b\right\}\right] \\
&= m^n (b-a)^{-mn}\left[\prod_{i=1}^{n}(y_i-a)\right]^{m-1}\prod_{i=1}^{n} I\left\{a<y_i<b\right\} \\
&= m^n (b-a)^{-mn}\left[\prod_{i=1}^{n}(y_i-a)\right]^{m-1} I\left\{\min y_i>a, \max y_i<b\right\}
\end{aligned}
$$

and

$$
\log L(a,b) = n\log m - mn\log(b-a) + (m-1)\sum_{i=1}^{n}\log\left(y_i-a\right) + \sum_{i=1}^{n}\log I\left\{\min y_i>a, \max y_i<b\right\}.
$$

The likelihood function is a decreasing function of b, so it MLE is $\max y_i$. The partial derivative of the log-likelihood with respect to a is

$$
\frac{\partial \log L(a,b)}{\partial a} = \frac{mn}{b-a} - (m-1)\sum_{i=1}^{n}\frac{1}{y_i-a}.
$$

So, the MLE of a is the root of

$$
\frac{mn}{b-a} = (m-1)\sum_{i=1}^{n}\frac{1}{y_i-a}.
$$

(6 marks)

UNSEEN