

**SOLUTIONS TO  
MATH68181  
EXTREME VALUES EXAM**

**Solutions to Question 1** a) The ARCH ( $q$ ) model is given by

$$e_t = \sigma_t Z_t$$

and

$$\sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 + \cdots + \alpha_q e_{t-q}^2.$$

b) The GARCH ( $p, q$ ) model given by

$$e_t = \sigma_t Z_t$$

and

$$\sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 + \cdots + e_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2.$$

c) The NGARCH model given by

$$e_t = \sigma_t Z_t$$

and

$$\sigma_t^2 = \omega + \alpha (e_{t-1} - \theta \sigma_{t-1})^2 + \beta \sigma_{t-1}^2.$$

d) Since

$$\begin{aligned} E(e_t) &= E(\sigma_t Z_t) \\ &= E[E(\sigma_t Z_t | \sigma_t)] \\ &= E[\sigma_t E(Z_t | \sigma_t)] \\ &= E[\sigma_t \cdot 0] \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} E(e_t^2) &= E(\sigma_t^2 Z_t^2) \\ &= E[E(\sigma_t^2 Z_t^2 | \sigma_t)] \\ &= E[\sigma_t^2 E(Z_t^2 | \sigma_t)] \\ &= E[\sigma_t^2], \end{aligned}$$

we the mean equal to zero and the variance equal to  $E[\sigma_t^2]$ .

e) Since stationarity holds, we have

$$\begin{aligned} E(\sigma_t^2) &= \alpha_0 + \alpha_1 E(e_{t-1}^2) + \cdots + \alpha_q E(e_{t-q}^2) \\ \Leftrightarrow E(\sigma_t^2) &= \alpha_0 + \alpha_1 E[E(e_{t-1}^2 | \sigma_{t-1})] + \cdots + \alpha_q E[E(e_{t-q}^2 | \sigma_{t-q})] \\ \Leftrightarrow E(\sigma_t^2) &= \alpha_0 + \alpha_1 E[\sigma_{t-1}^2 E(Z_{t-1}^2 | \sigma_{t-1})] + \cdots + \alpha_q E[\sigma_{t-q}^2 E(Z_{t-q}^2 | \sigma_{t-q})] \\ \Leftrightarrow E(\sigma_t^2) &= \alpha_0 + \alpha_1 E[\sigma_{t-1}^2 \cdot 1] + \cdots + \alpha_q E[\sigma_{t-q}^2 \cdot 1] \\ \Leftrightarrow E(\sigma_t^2) &= \alpha_0 + \alpha_1 E[\sigma_{t-1}^2] + \cdots + \alpha_q E[\sigma_{t-q}^2] \\ \Leftrightarrow \sigma^2 &= \alpha_0 + \alpha_1 \sigma^2 + \cdots + \alpha_q \sigma^2 \\ \Leftrightarrow \sigma^2 &= \frac{\alpha_0}{1 - \alpha_1 - \cdots - \alpha_q}. \end{aligned}$$

f) Since stationarity holds, we have

$$\begin{aligned}
& E(\sigma_t^2) = \alpha_0 + \alpha_1 E(e_{t-1}^2) + \dots + \alpha_q E(e_{t-q}^2) \\
& \quad + \beta_1 E(\sigma_{t-1}^2) + \dots + \beta_p E(\sigma_{t-p}^2) \\
\Leftrightarrow & E(\sigma_t^2) = \alpha_0 + \alpha_1 E[E(e_{t-1}^2 | \sigma_{t-1})] + \dots + \alpha_q E[E(e_{t-q}^2 | \sigma_{t-q})] \\
& \quad + \beta_1 E(\sigma_{t-1}^2) + \dots + \beta_p E(\sigma_{t-p}^2) \\
\Leftrightarrow & E(\sigma_t^2) = \alpha_0 + \alpha_1 E[\sigma_{t-1}^2 E(Z_{t-1}^2 | \sigma_{t-1})] + \dots + \alpha_q E[\sigma_{t-q}^2 E(Z_{t-q}^2 | \sigma_{t-q})] \\
& \quad + \beta_1 E(\sigma_{t-1}^2) + \dots + \beta_p E(\sigma_{t-p}^2) \\
\Leftrightarrow & E(\sigma_t^2) = \alpha_0 + \alpha_1 E[\sigma_{t-1}^2 \cdot 1] + \dots + \alpha_q E[\sigma_{t-q}^2 \cdot 1] + \beta_1 E(\sigma_{t-1}^2) + \dots + \beta_p E(\sigma_{t-p}^2) \\
\Leftrightarrow & E(\sigma_t^2) = \alpha_0 + \alpha_1 E[\sigma_{t-1}^2] + \dots + \alpha_q E[\sigma_{t-q}^2] + \beta_1 E(\sigma_{t-1}^2) + \dots + \beta_p E(\sigma_{t-p}^2) \\
\Leftrightarrow & \sigma^2 = \alpha_0 + \alpha_1 \sigma^2 + \dots + \alpha_q \sigma^2 + \beta_1 \sigma^2 + \dots + \beta_p \sigma^2 \\
\Leftrightarrow & \sigma^2 = \frac{\alpha_0}{1 - \alpha_1 - \dots - \alpha_q - \beta_1 - \dots - \beta_p}.
\end{aligned}$$

g) Since stationarity holds, we have

$$\begin{aligned}
& E(\sigma_t^2) = \omega + \alpha E[(e_{t-1} - \theta \sigma_{t-1})^2] + \beta E(\sigma_{t-1}^2) \\
\Leftrightarrow & E(\sigma_t^2) = \omega + \alpha E(e_{t-1}^2) - 2\alpha\theta E(e_{t-1}\sigma_{t-1}) + \alpha\theta^2 E(\sigma_{t-1}^2) + \beta E(\sigma_{t-1}^2) \\
\Leftrightarrow & \sigma^2 = \omega + \alpha E[E(e_{t-1}^2 | \sigma_{t-1})] - 2\alpha\theta E[E(e_{t-1}\sigma_{t-1} | \sigma_{t-1})] + \alpha\theta^2 \sigma^2 + \beta \sigma^2 \\
\Leftrightarrow & \sigma^2 = \omega + \alpha E[\sigma_{t-1}^2 E(Z_{t-1}^2 | \sigma_{t-1})] - 2\alpha\theta E[\sigma_{t-1}^2 E(Z_{t-1} | \sigma_{t-1})] + \alpha\theta^2 \sigma^2 + \beta \sigma^2 \\
\Leftrightarrow & \sigma^2 = \omega + \alpha E[\sigma_{t-1}^2 \cdot 1] - 2\alpha\theta E[\sigma_{t-1}^2 \cdot 0] + \alpha\theta^2 \sigma^2 + \beta \sigma^2 \\
\Leftrightarrow & \sigma^2 = \omega + \alpha E[\sigma_{t-1}^2] + \alpha\theta^2 \sigma^2 + \beta \sigma^2 \\
\Leftrightarrow & \sigma^2 = \omega + \alpha \sigma^2 + \alpha\theta^2 \sigma^2 + \beta \sigma^2 \\
\Leftrightarrow & \sigma^2 = \frac{\omega}{1 - \alpha - \alpha\theta^2 - \beta}.
\end{aligned}$$

## Solutions to Question 2

$C(u_1, u_2)$  is a valid copula if

$$C(u, 0) = 0,$$

$$C(0, u) = 0,$$

$$C(1, u) = u,$$

$$C(u, 1) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) \geq 0.$$

a) for the copula defined by  $C(u_1, u_2) = \min(u_1, u_2)$ , we have

$$C(u, 0) = \min(u, 0) = 0,$$

$$C(0, u) = \min(0, u) = 0,$$

$$C(1, u) = \min(1, u) = u,$$

$$C(u, 1) = \min(u, 1) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \begin{cases} 1, & \text{if } u_1 \leq u_2, \\ 0, & \text{if } u_1 > u_2, \end{cases}$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = \begin{cases} 0, & \text{if } u_1 \leq u_2, \\ 1, & \text{if } u_1 > u_2, \end{cases}$$

so  $C$  is a valid copula.

b) for the copula defined by  $C(u_1, u_2) = u_1 u_2 \exp[-\theta \log u_1 \log u_2]$ , we have

$$C(u, 0) = u \cdot 0 \cdot \exp[-\theta \log 0 \log u] = 0,$$

$$C(0, u) = 0 \cdot u \exp[-\theta \log u \log 0] = 0,$$

$$C(1, u) = 1 \cdot u \exp[-\theta \log 1 \log u] = u,$$

$$C(u, 1) = u \cdot 1 \exp[-\theta \log u \log 1] = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = u_2 (1 - \theta \log u_2) \exp[-\theta \log u_1 \log u_2] \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = u_1 (1 - \theta \log u_1) \exp[-\theta \log u_1 \log u_2] \geq 0,$$

so  $C$  is a valid copula.

c) for the Farlie-Gumbel-Morgenstern copula defined by

$$C(u_1, u_2) = u_1 u_2 [1 + \phi(1 - u_1)(1 - u_2)],$$

we have

$$C(u, 0) = u \cdot 0 [1 + \phi(1 - u)(1 - 0)] = 0,$$

$$C(0, u) = 0 \cdot u [1 + \phi(1 - 0)(1 - u)] = 0,$$

$$C(u, 1) = u \cdot 1 [1 + \phi(1 - u)(1 - 1)] = u,$$

$$C(1, u) = 1 \cdot u [1 + \phi(1 - 1)(1 - u)] = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = u_2 [1 + \phi(1 - u_1)(1 - u_2)] + u_1 u_2 [1 - \phi(1 - u_2)] \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) = u_1 [1 + \phi(1 - u_1)(1 - u_2)] + u_1 u_2 [1 - \phi(1 - u_1)] \geq 0$$

so  $C$  is a valid copula.

d) for the Burr copula defined by  $C(u_1, u_2) = u_1 + u_2 - 1 + \left[ (1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1 \right]^{-\alpha}$ , we have

$$C(u, 0) = u + 0 - 1 + \left[ (1 - u)^{-1/\alpha} + (1 - 0)^{-1/\alpha} - 1 \right]^{-\alpha} = 0,$$

$$C(0, u) = 0 + u - 1 + \left[ (1 - 0)^{-1/\alpha} + (1 - u)^{-1/\alpha} - 1 \right]^{-\alpha} = 0,$$

$$C(u, 1) = u + 1 - 1 + \left[ (1 - u)^{-1/\alpha} + (1 - 1)^{-1/\alpha} - 1 \right]^{-\alpha} = 1,$$

$$C(1, u) = 1 + u - 1 + \left[ (1 - 1)^{-1/\alpha} + (1 - u)^{-1/\alpha} - 1 \right]^{-\alpha} = 1,$$

$$\begin{aligned} \frac{\partial}{\partial u_1} C(u_1, u_2) &= 1 - (1 - u_1)^{-1/\alpha - 1} \left[ (1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1 \right]^{-\alpha - 1} \\ &= 1 - \left[ \frac{(1 - u_1)^{-1/\alpha}}{(1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1} \right]^{\alpha + 1} \geq 0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial u_2} C(u_1, u_2) &= 1 - (1 - u_2)^{-1/\alpha - 1} \left[ (1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1 \right]^{-\alpha - 1} \\ &= 1 - \left[ \frac{(1 - u_2)^{-1/\alpha}}{(1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1} \right]^{\alpha + 1} \geq 0, \end{aligned}$$

so  $C$  is a valid copula.

### Solutions to Question 3

a) We can write

$$\bar{F}(x, y) = \exp \left[ - (x^a + y^a)^{1/a} \right] = \exp \left\{ - (x + y) \left[ \left( \frac{y}{x + y} \right)^a + \left( \frac{x}{x + y} \right)^a \right] \right\}.$$

This is in the form of

$$\bar{F}(x, y) = \exp \left[ - (x + y) A \left( \frac{y}{x + y} \right) \right]$$

with  $A(t) = [t^a + (1 - t)^a]^{1/a}$ .

We now check the conditions for  $A(\cdot)$ . Clearly,  $A(0) = 1$  and  $A(1) = 1$ .

Also  $A(t) \geq 0$  since  $t^a \geq 0$  and  $(1 - t)^a \geq 0$  for all  $t$ .

To show that  $A(t) \leq 1$ , note that

$$\begin{aligned} A(t) &\leq 1 \\ \Leftrightarrow [t^a + (1 - t)^a]^{1/a} &\leq 1 \\ \Leftrightarrow t^a + (1 - t)^a &\leq 1. \end{aligned}$$

Now let  $g(t) = t^a + (1 - t)^a$ . We have  $g'(t) = at^{a-1} - a(1 - t)^{a-1}$ ,  $g'(0) = -a$ ,  $g'(1) = a$  and  $g''(t) = a(a - 1)t^{a-2} + a(a - 1)(1 - t)^{a-2}$ . So,  $g(t)$  attains its maximum at  $t = 0$  or  $t = 1$ . Hence,  $t^a + (1 - t)^a \leq 1$  holds for all  $t$ .

Also  $A(t) \geq t$  since

$$[t^a + (1 - t)^a]^{1/a} \geq [t^a]^{1/a} \geq t.$$

Also  $A(t) \geq 1 - t$  since

$$[t^a + (1 - t)^a]^{1/a} \geq [(1 - t)^a]^{1/a} \geq 1 - t.$$

$A(\cdot)$  is convex since

$$A'(t) = [t^a + (1 - t)^a]^{1/a-1} [t^{a-1} - (1 - t)^{a-1}]$$

and

$$A''(t) = (a - 1) [t^a + (1 - t)^a]^{1/a-2} [t^a(1 - t)^{a-2} + t^{a-2}(1 - t)^a + 2t^{a-1}(1 - t)^{a-1}] \geq 0.$$

b) the joint cdf is

$$F(x, y) = 1 - \exp(-x) - \exp(-y) + \exp \left[ - (x^a + y^a)^{1/a} \right].$$

c) the derivative of joint cdf with respect to  $x$  is

$$\frac{\partial F(x, y)}{\partial x} = \exp(-x) - x^{a-1} (x^a + y^a)^{1/a-1} \exp \left[ - (x^a + y^a)^{1/a} \right],$$

so the conditional cdf if  $Y$  given  $X = x$  is

$$F(y|x) = 1 - x^{a-1} (x^a + y^a)^{1/a-1} \exp \left[ x - (x^a + y^a)^{1/a} \right].$$

d) the derivative of joint cdf with respect to  $y$  is

$$\frac{\partial F(x, y)}{\partial y} = \exp(-y) - y^{a-1} (x^a + y^a)^{1/a-1} \exp \left[ - (x^a + y^a)^{1/a} \right],$$

so the conditional cdf if  $X$  given  $Y = y$  is

$$F(x|y) = 1 - y^{a-1} (x^a + y^a)^{1/a-1} \exp \left[ y - (x^a + y^a)^{1/a} \right].$$

e) the derivative of joint cdf with respect to  $x$  and  $y$  is

$$\begin{aligned} f(x, y) &= \frac{\partial F(x, y)}{\partial x \partial y} \\ &= (xy)^{a-1} (x^a + y^a)^{1/a-2} \exp \left[ - (x^a + y^a)^{1/a} \right] \\ &\quad \cdot \left[ a - 1 + (x^a + y^a)^{1/a} \right]. \end{aligned}$$



### Solutions to Question 4

a) Let  $X$  denote the actual stock return. The pdf of  $X$  is

$$\begin{aligned}
 f_X(x) &= \frac{1}{b-a} \int_a^b \lambda \exp(-\lambda x) d\lambda \\
 &= \frac{1}{b-a} \left\{ \left[ \lambda \frac{\exp(-\lambda x)}{-x} \right]_a^b + \frac{1}{x} \int_a^b \exp(-\lambda x) d\lambda \right\} \\
 &= \frac{1}{b-a} \left\{ -\frac{b \exp(-bx) - a \exp(-ax)}{x} - \frac{\exp(-bx) - \exp(-ax)}{x^2} \right\} \\
 &= \frac{(xa+1) \exp(-ax) - (xb+1) \exp(-bx)}{x^2(b-a)}.
 \end{aligned}$$

b) the expected value of  $X$  is

$$\begin{aligned}
 E(X) &= \int_0^\infty \frac{(xa+1) \exp(-ax) - (xb+1) \exp(-bx)}{x(b-a)} dx \\
 &= \frac{1}{b-a} \left[ a \int_0^\infty \exp(-ax) dx - b \int_0^\infty \exp(-bx) dx + \int_0^\infty \frac{1}{x} \exp(-ax) dx - \int_0^\infty \frac{1}{x} \exp(-bx) dx \right] \\
 &= \frac{1}{b-a} \left[ 1 - 1 + \int_0^\infty \frac{1}{x} \exp(-ax) dx - \int_0^\infty \frac{1}{x} \exp(-bx) dx \right] \\
 &= \frac{1}{b-a} \left[ \int_0^\infty \frac{1}{x} \exp(-ax) dx - \int_0^\infty \frac{1}{x} \exp(-bx) dx \right] \\
 &= \frac{1}{b-a} [\infty - \infty] \\
 &= \infty.
 \end{aligned}$$

c) the expected value of  $X^2$  is

$$\begin{aligned}
 E(X^2) &= \int_0^\infty \frac{(xa+1) \exp(-ax) - (xb+1) \exp(-bx)}{b-a} dx \\
 &= \frac{1}{b-a} \left[ a \int_0^\infty x \exp(-ax) dx - b \int_0^\infty x \exp(-bx) dx + \int_0^\infty \exp(-ax) dx - \int_0^\infty \exp(-bx) dx \right] \\
 &= \frac{1}{b-a} \left[ \frac{1}{a} - \frac{1}{b} + \frac{1}{a} - \frac{1}{b} \right] \\
 &= \frac{2}{ab}.
 \end{aligned}$$

Hence, the variance is infinite.

d) If  $x_1, x_2, \dots, x_n$  is a random sample on  $X$  then the likelihood function is

$$L(a, b) = (b-a)^{-n} \prod_{i=1}^n \frac{(x_i a + 1) \exp(-a x_i) - (x_i b + 1) \exp(-b x_i)}{x_i^2}.$$

The log-likelihood function is

$$\log L(a, b) = -n \log(b - a) + \sum_{i=1}^n \log [(x_i a + 1) \exp(-ax_i) - (x_i b + 1) \exp(-bx_i)] - 2 \sum_{i=1}^n \log x_i.$$

The partial derivatives with respect to  $a$  and  $b$  are

$$\frac{\partial \log L}{\partial a} = \frac{n}{b - a} - a \sum_{i=1}^n \frac{x_i^2 \exp(-ax_i)}{(x_i a + 1) \exp(-ax_i) - (x_i b + 1) \exp(-bx_i)}$$

and

$$\frac{\partial \log L}{\partial b} = -\frac{n}{b - a} - b \sum_{i=1}^n \frac{x_i^2 \exp(-bx_i)}{(x_i a + 1) \exp(-ax_i) - (x_i b + 1) \exp(-bx_i)}.$$

So, the mles of  $a$  and  $b$  are the simultaneous solutions of the equations

$$\frac{n}{b - a} = a \sum_{i=1}^n \frac{x_i^2 \exp(-ax_i)}{(x_i a + 1) \exp(-ax_i) - (x_i b + 1) \exp(-bx_i)}$$

and

$$-\frac{n}{b - a} = b \sum_{i=1}^n \frac{x_i^2 \exp(-bx_i)}{(x_i a + 1) \exp(-ax_i) - (x_i b + 1) \exp(-bx_i)}.$$

## Solutions to Question 5

If there are norming constants  $a_n > 0$ ,  $b_n$  and a nondegenerate  $G$  such that the cdf of a normalized version of  $M_n$  converges to  $G$ , i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n(a_n x + b_n) \rightarrow G(x) \quad (1)$$

as  $n \rightarrow \infty$  then  $G$  must be of the same type as (cdf's  $G$  and  $G^*$  are of the same type if  $G^*(x) = G(ax + b)$  for some  $a > 0$ ,  $b$  and all  $x$ ) as one of the following three classes:

$$\begin{aligned} I & : \Lambda(x) = \exp\{-\exp(-x)\}, \quad x \in \mathfrak{R}; \\ II & : \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\{-x^{-\alpha}\} & \text{if } x \geq 0 \end{cases} \\ & \quad \text{for some } \alpha > 0; \\ III & : \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^\alpha\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \\ & \quad \text{for some } \alpha > 0. \end{aligned}$$

The necessary and sufficient conditions for the three extreme value distributions are:

$$\begin{aligned} I & : \exists \gamma(t) > 0 \text{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = \exp(-x), \quad x \in \mathfrak{R}, \\ II & : w(F) = \infty \text{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \quad x > 0, \\ III & : w(F) < \infty \text{ and } \lim_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} = x^\alpha, \quad x > 0. \end{aligned}$$

Firstly, suppose that  $G$  belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function, say  $h(t)$ , such that

$$\lim_{t \rightarrow w(G)} \frac{1 - G(t + xh(t))}{1 - G(t)} = \exp(-x)$$

for every  $x \in (-\infty, \infty)$ . But, using L'Hopital's rule, we note that

$$\begin{aligned} \lim_{t \rightarrow w(F)} \frac{1 - F(t + xh(t))}{1 - F(t)} & = \lim_{t \rightarrow w(F)} \frac{[1 + xh'(t)]f(t + xh(t))}{f(t)} \\ & = \lim_{t \rightarrow w(G)} \frac{[1 + xh'(t)]g(t + xh(t))}{g(t)} \left[ \frac{G(t + xh(t))}{G(t)} \right]^{a-1} \\ & \quad \times \left[ \frac{1 - G(t + xh(t))}{1 - G(t)} \right]^{b-1} \exp\{cG(t) - cG(t + xh(t))\} \\ & = \exp(-bx) \end{aligned}$$

for every  $x \in (-\infty, \infty)$ . So, it follows that  $F$  also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n \rightarrow \infty} \Pr \{a_n (M_n - b_n) \leq x\} = \exp \{-\exp(-bx)\}$$

for some suitable norming constants  $a_n > 0$  and  $b_n$ .

Secondly, suppose that  $G$  belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a  $\beta < 0$  such that

$$\lim_{t \rightarrow \infty} \frac{1 - G(tx)}{1 - G(t)} = x^\beta$$

for every  $x > 0$ . But, using L'Hopital's rule, we note that

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1 - F(tx)}{1 - F(t)} &= \lim_{t \rightarrow \infty} \frac{xf(tx)}{f(t)} \\ &= \lim_{t \rightarrow \infty} \frac{xg(tx)}{g(t)} \left[ \frac{G(tx)}{G(t)} \right]^{a-1} \left[ \frac{1 - G(tx)}{1 - G(t)} \right]^{b-1} \exp \{cG(t) - cG(tx)\} \\ &= \lim_{t \rightarrow \infty} \frac{xg(tx)}{g(t)} \left[ \frac{1 - G(tx)}{1 - G(t)} \right]^{b-1} \\ &= \lim_{t \rightarrow \infty} \frac{1 - G(tx)}{1 - G(t)} \left[ \frac{1 - G(tx)}{1 - G(t)} \right]^{b-1} \\ &= \lim_{t \rightarrow \infty} \left[ \frac{1 - G(tx)}{1 - G(t)} \right]^b \\ &= x^{b\beta} \end{aligned}$$

for every  $x > 0$ . So, it follows that  $F$  also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n \rightarrow \infty} \Pr \{a_n (M_n - b_n) \leq x\} = \exp(-x^{b\beta})$$

for some suitable norming constants  $a_n > 0$  and  $b_n$ .

Thirdly, suppose that  $G$  belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a  $\alpha > 0$  such that

$$\lim_{t \rightarrow 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} = x^\alpha$$

for every  $x > 0$ . But, using L'Hopital's rule, we note that

$$\begin{aligned}
\lim_{t \rightarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} &= \lim_{t \rightarrow 0} \frac{xf(w(F) - tx)}{f(w(F) - t)} \\
&= \lim_{t \rightarrow 0} \frac{xg(w(F) - tx)}{g(w(F) - t)} \left[ \frac{G(w(F) - tx)}{G(w(F) - t)} \right]^{a-1} \left[ \frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \right]^{b-1} \\
&\quad \times \exp \{cG(w(F) - t) - cG(w(F) - tx)\} \\
&= \lim_{t \rightarrow 0} \frac{xg(w(F) - tx)}{g(w(F) - t)} \left[ \frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \right]^{b-1} \\
&= \lim_{t \rightarrow 0} \frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \left[ \frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \right]^{b-1} \\
&= \lim_{t \rightarrow 0} \left[ \frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \right]^b \\
&= x^{b\alpha}.
\end{aligned}$$

So, it follows that  $F$  also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{n \rightarrow \infty} \Pr \{a_n (M_n - b_n) \leq x\} = \exp \{ -(-x)^{b\alpha} \}$$

for some suitable norming constants  $a_n > 0$  and  $b_n$ .

## Solutions to Question 6

a) Note that  $w(F) = \infty$ . Then

$$\begin{aligned}
 \lim_{t \uparrow \infty} \frac{1 - F(t + xg(t))}{1 - F(t)} &= \lim_{t \uparrow \infty} \frac{1 - \{1 - \exp[1 - (1 + \lambda t + \lambda xg(t))^\alpha]\}}{1 - \{1 - \exp[1 - (1 + \lambda t)^\alpha]\}} \\
 &= \lim_{t \uparrow \infty} \frac{\exp[1 - (1 + \lambda t + \lambda xg(t))^\alpha]}{\exp[1 - (1 + \lambda t)^\alpha]} \\
 &= \lim_{t \uparrow \infty} \exp[(1 + \lambda t)^\alpha - (1 + \lambda t + \lambda xg(t))^\alpha] \\
 &= \lim_{t \uparrow \infty} \exp \left\{ (1 + \lambda t)^\alpha \left[ 1 - \left( 1 + \frac{\lambda g(t)x}{1 + \lambda t} \right)^\alpha \right] \right\} \\
 &= \lim_{t \uparrow \infty} \exp \left\{ (1 + \lambda t)^\alpha \left[ 1 - \left( 1 + \alpha \frac{\lambda g(t)x}{1 + \lambda t} \right) \right] \right\} \quad \text{using } (1 + x)^a \approx 1 + ax \\
 &= \lim_{t \uparrow \infty} \exp \left\{ -(1 + \lambda t)^\alpha \left[ \alpha \frac{\lambda g(t)x}{1 + \lambda t} \right] \right\} \\
 &= \lim_{t \uparrow \infty} \exp \left\{ -\lambda \alpha (1 + \lambda t)^{\alpha-1} g(t)x \right\} \\
 &= \exp \{-x\}
 \end{aligned}$$

if  $g(t) = 1/(\lambda\alpha)(1+\lambda t)^{1-\alpha}$ . So, the exponentiated extension cdf  $F(x) = 1 - \exp[1 - (1 + \lambda x)^\alpha]^\alpha$  belongs to the Gumbel domain of attraction.

b) Note that  $w(F) = \infty$ . Then

$$\begin{aligned}
 \lim_{t \rightarrow \infty} \frac{1 - F(tx)}{1 - F(t)} &= \lim_{t \rightarrow \infty} \frac{[1 - \exp(-\frac{\lambda}{tx})]^\alpha}{[1 - \exp(-\frac{\lambda}{t})]^\alpha} \\
 &= \lim_{t \rightarrow \infty} \left[ \frac{1 - \exp(-\frac{\lambda}{tx})}{1 - \exp(-\frac{\lambda}{t})} \right]^\alpha \\
 &= \lim_{t \rightarrow \infty} \left[ \frac{1 - (1 - \frac{\lambda}{tx})}{1 - (1 - \frac{\lambda}{t})} \right]^\alpha \quad \text{using } \exp(-a) \approx 1 - a \\
 &= \lim_{t \rightarrow \infty} \left[ \frac{\frac{\lambda}{tx}}{\frac{\lambda}{t}} \right]^\alpha \\
 &= x^{-\alpha}.
 \end{aligned}$$

So, the inverse exponentiated exponential cdf  $F(x) = 1 - [1 - \exp(-\frac{\lambda}{x})]^\alpha$  belongs to the Fréchet domain of attraction.

c) For the Poisson distribution,

$$\frac{\Pr(X = k)}{1 - F(k-1)} = \frac{\lambda^k/k!}{\sum_{j=k}^{\infty} \lambda^j/j!} = \frac{1}{1 + \sum_{j=k+1}^{\infty} k! \lambda^{j-k}/j!}.$$

The term in the denominator can be rewritten as

$$\sum_{j=1}^{\infty} \frac{\lambda^j}{(k+1)(k+2)\cdots(k+j)} \leq \sum_{j=1}^{\infty} \left(\frac{\lambda}{k}\right)^j = \frac{\lambda/k}{1-\lambda/k}$$

(when  $k > \lambda$ ) and the bound tends to 0 as  $k \rightarrow \infty$  and so it follows that  $p(k)/(1-F(k-1)) \rightarrow 1$ . Hence, there can be no non-degenerate limit.

d) For the Bernoulli ( $p$ ) distribution,

$$\frac{\Pr(X = k)}{1 - F(k-1)} = \begin{cases} 1 - p, & \text{if } k = 0, \\ 1, & \text{if } k = 1. \end{cases}$$

Hence, there can be no sequences  $a_n > 0$  and  $b_n$  such that  $(M_n - b_n)/a_n$  has a non-degenerate limiting distribution.

e) For the discrete Weibull distribution, the corresponding pmf is

$$p(x) = q^{x^a} - q^{(x+1)^a}.$$

So,

$$\begin{aligned} \frac{\Pr(X = x)}{1 - F(x-1)} &= \frac{q^{x^a} - q^{(x+1)^a}}{1 - [1 - q^{x^a}]} \\ &= \frac{q^{x^a} - q^{(x+1)^a}}{q^{x^a}} \\ &= 1 - q^{(x+1)^a - x^a}. \end{aligned}$$

Note that

$$\begin{aligned} x^a - (x+1)^a &= x^a - x^a \left(1 + \frac{1}{x}\right)^a \\ &= x^a \left[1 - \left(1 + \frac{1}{x}\right)^a\right] \\ &= x^a \left[1 - 1 - a\frac{1}{x} - \frac{a(a-1)}{2!} \frac{1}{x^2} - \dots\right] \\ &\rightarrow -\infty. \end{aligned}$$

Hence,

$$\frac{\Pr(X = x)}{1 - F(x-1)} \rightarrow 1.$$

Hence, there can be no sequences  $a_n > 0$  and  $b_n$  such that  $(M_n - b_n)/a_n$  has a non-degenerate limiting distribution.

### Solutions to Question 7

If  $X$  is an absolutely continuous random variable with cdf  $F(\cdot)$  then

$$\text{VaR}_p(X) = F^{-1}(p)$$

and

$$\text{ES}_p(X) = \frac{1}{p} \int_0^p F^{-1}(v) dv.$$

Setting

$$\Phi\left(\frac{x - \mu}{\sigma}\right) = p$$

gives

$$\text{VaR}_p(X) = \mu + \sigma \Phi^{-1}(p)$$

and

$$\text{ES}_p(X) = \mu + \frac{\sigma}{p} \int_0^p \Phi^{-1}(v) dv.$$

a) The joint likelihood function of  $\mu$  and  $\sigma^2$  is

$$\begin{aligned} L(\mu, \sigma^2) &= \prod_{i=1}^n \left\{ \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(X_i - \mu)^2}{2\sigma^2}\right] \right\} \\ &= \frac{1}{(2\pi)^{n/2} \sigma^n} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^n (X_i - \mu)^2\right]. \end{aligned}$$

The joint log likelihood function of  $\mu$  and  $\sigma^2$  is

$$\log L(\mu, \sigma^2) = -\frac{n}{2} \log(2\pi) - n \log \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^n (X_i - \mu)^2.$$

The first order partial derivatives of this with respect to  $\mu$  and  $\sigma$  are

$$\frac{\partial \log L}{\partial \mu} = \frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \mu) = \frac{1}{\sigma^2} \left( \sum_{i=1}^n X_i - n\mu \right) \quad (2)$$

and

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^n (X_i - \mu)^2, \quad (3)$$



respectively.

b) Using equation (2), one can see that the solution of  $\partial \log L / \partial \mu = 0$  is  $\mu = \bar{X} = (1/n) \sum_{i=1}^n X_i$ .

c) Using equation (3), one can see that the solution of  $\partial \log L / \partial \sigma = 0$  is  $\sigma^2 = S^2 = (1/n) \sum_{i=1}^n (X_i - \bar{X})^2$ .

d) The mle of Value at Risk is

$$\widehat{\text{VaR}}_p(X) = \bar{X} + S\Phi^{-1}(p)$$

The mle of Expected Shortfall is

$$\widehat{\text{ES}}_p(X) = \bar{X} + \frac{S}{p} \int_0^p \Phi^{-1}(v) dv.$$

e) Since

$$\begin{aligned} E(\bar{X}) &= E\left(\frac{1}{n} \sum_{i=1}^n X_i\right) \\ &= \frac{1}{n} \sum_{i=1}^n E(X_i) \\ &= \frac{1}{n} \sum_{i=1}^n \mu \\ &= \mu, \end{aligned}$$

$\bar{X}$  is unbiased for  $\mu$ . Since  $\sum_{i=1}^n (X_i - \bar{X})^2 \sim \sigma^2 \chi_{n-1}^2$  and  $E(\chi_k) = \sqrt{2}\Gamma((k+1)/2) / \Gamma(k/2)$ , we can write

$$\begin{aligned} E(S) &= E\left[\frac{\sigma}{\sqrt{n}} \sqrt{\chi_{n-1}^2}\right] \\ &= \frac{\sigma}{\sqrt{n}} E\left[\sqrt{\chi_{n-1}^2}\right] \\ &= \frac{\sigma}{\sqrt{n}} \frac{\sqrt{2}\Gamma(n/2)}{\Gamma((n-1)/2)}, \end{aligned}$$

so  $S$  is biased for  $\sigma$ .

Since

$$\begin{aligned} E\left(\widehat{\text{VaR}}_p(X)\right) &= E(\bar{X}) + E(S)\Phi^{-1}(p) \\ &= \mu + \frac{\sigma}{\sqrt{n}} \frac{\sqrt{2}\Gamma(n/2)}{\Gamma((n-1)/2)} \Phi^{-1}(p) \\ &\neq \mu + \sigma \Phi^{-1}(p), \end{aligned}$$

$\widehat{\text{VaR}}_p(X)$  is biased for  $\text{VaR}_p(X)$ .

f) Since

$$\begin{aligned} E\left(\widehat{\text{ES}}_p(X)\right) &= E(\bar{X}) + E(S) \frac{1}{p} \int_0^p \Phi^{-1}(v) dv \\ &= \mu + \frac{\sigma}{\sqrt{n}} \frac{\sqrt{2}\Gamma(n/2)}{\Gamma((n-1)/2)} \frac{1}{p} \int_0^p \Phi^{-1}(v) dv \\ &\neq \mu + \sigma \frac{1}{p} \int_0^p \Phi^{-1}(v) dv, \end{aligned}$$

$\widehat{\text{ES}}_p(X)$  is biased for  $\text{ES}_p(X)$ .

### Solutions to Question 8

a) The cdf of  $X$  is

$$\begin{aligned}F_Y(y) &= \Pr(Y \leq y) \\&= \Pr(\min(X_1, \dots, X_\alpha) \leq y) \\&= 1 - \Pr(\min(X_1, \dots, X_\alpha) > y) \\&= 1 - \Pr(X_1 > y, \dots, X_\alpha > y) \\&= 1 - \Pr(X_1 > y) \cdots \Pr(X_\alpha > y) \\&= 1 - \exp(-\lambda y) \cdots \exp(-\lambda y) \\&= 1 - \exp(-\alpha \lambda y),\end{aligned}$$

the exponential cdf with parameter  $\alpha\lambda$ .

b) The corresponding pdf is

$$f_Y(y) = \alpha\lambda \exp(-\alpha\lambda y).$$

c) The  $n$ th moment of  $Y$  can be calculated as

$$\begin{aligned}E(Y^n) &= \alpha\lambda \int_0^\infty x^n \exp(-\alpha\lambda x) dx \\&= (\alpha\lambda)^{-n} \int_0^\infty x^n \exp(-x) dx \\&= (\alpha\lambda)^{-n} \Gamma(n+1) \\&= (\alpha\lambda)^{-n} n!.\end{aligned}$$

So,

$$E(Y) = (\alpha\lambda)^{-1}$$

and

$$Var(Y) = (\alpha\lambda)^{-2}.$$

d) Setting

$$1 - \exp(-\alpha\lambda y) = p$$

gives

$$\text{VaR}_p(Y) = -\frac{1}{\alpha\lambda} \log(1-p).$$

e) The expected shortfall is

$$\begin{aligned}
\text{ES}_p(Y) &= -\frac{1}{\alpha\lambda p} \int_0^p \log(1-v) \, dv \\
&= -\frac{1}{\alpha\lambda p} \left\{ [v \log(1-v)]_0^p + \int_0^p \frac{v}{1-v} \, dv \right\} \\
&= -\frac{1}{\alpha\lambda p} \left\{ p \log(1-p) + \int_0^p \frac{v-1+1}{1-v} \, dv \right\} \\
&= -\frac{1}{\alpha\lambda p} \left\{ p \log(1-p) - p + \int_0^p \frac{1}{1-v} \, dv \right\} \\
&= -\frac{1}{\alpha\lambda p} \{ p \log(1-p) - p - \log(1-p) \}.
\end{aligned}$$

f) The likelihood function is

$$L(\alpha, \lambda) = \alpha^n \lambda^n \exp\left(-\alpha\lambda \sum_{i=1}^n y_i\right).$$

The log-likelihood function is

$$\log L = n \log(\alpha\lambda) - \alpha\lambda \sum_{i=1}^n y_i.$$

The partial derivatives with respect to  $\alpha$  and  $\lambda$  are

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} - \lambda \sum_{i=1}^n y_i$$

and

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \alpha \sum_{i=1}^n y_i.$$

Setting these to zero, we find that the mles of  $\alpha$  and  $\lambda$  are the solutions of

$$\hat{\alpha} = \frac{n}{\lambda \sum_{i=1}^n y_i}.$$

By definition,  $\alpha$  must be a positive integer. Hence, the set of all possible mles of  $\alpha$  and  $\lambda$  is

$$\left\{ \left( m, \frac{n}{m \sum_{i=1}^n y_i} \right), m = 1, 2, \dots \right\}.$$