SOLUTIONS TO MATH38181 EXTREME VALUES EXAM

Solutions to Question 1

a) We can write

$$\overline{F}(x,y) = \exp\left[-\left(x^a + y^a\right)^{1/a}\right] = \exp\left\{-(x+y)\left[\left(\frac{y}{x+y}\right)^a + \left(\frac{x}{x+y}\right)^a\right]\right\}.$$

This is in the form of

$$\overline{F}(x,y) = \exp\left[-(x+y)A\left(\frac{y}{x+y}\right)\right]$$

with $A(t) = [t^a + (1-t)^a]^{1/a}$.

We now check the conditions for $A(\cdot)$. Clearly, $A(0) = 1$ and $A(1) = 1$.

Also $A(t) \geq 0$ since $t^a \geq 0$ and $(1-t)^a \geq 0$ for all t.

To show that $A(t) \leq 1$, note that

$$\begin{aligned} & A(t) \leq 1 \\ \Leftrightarrow\ & [t^a + (1-t)^a]^{1/a} \leq 1 \\ \Leftrightarrow\ & t^a + (1-t)^a \leq 1. \end{aligned}$$

Now let $g(t) = t^a + (1-t)^a$. We have $g^{'}(t) = at^{a-1} - a(1-t)^{a-1}$, $g^{'}(0) = -a$, $g^{'}(1) = a$ and $g^{''}(t) = a(a-1)t^{a-2} + a(a-1)(1-t)^{a-2}$. So, $g(t)$ attains its maximum at $t = 0$ or $t = 1$. Hence, $t^a + (1-t)^a \leq 1$ holds for all t.

Also $A(t) \geq t$ since

$$[t^a + (1-t)^a]^{1/a} \geq [t^a]^{1/a} \geq t.$$

Also $A(t) \geq 1 - t$ since

$$[t^a + (1-t)^a]^{1/a} \geq [(1-t)^a]^{1/a} \geq 1 - t.$$

$A(\cdot)$ is convex since

$$A^{'}(t) = [t^a + (1-t)^a]^{1/a-1}\left[t^{a-1} - (1-t)^{a-1}\right]$$

and

$$A^{''}(t) = (a-1)\left[t^a + (1-t)^a\right]^{1/a-2}\left[t^a(1-t)^{a-2} + t^{a-2}(1-t)^a + 2t^{a-1}(1-t)^{a-1}\right] \geq 0.$$

b) the joint cdf is

$$F(x,y) = 1 - \exp(-x) - \exp(-y) + \exp\left[-\left(x^a + y^a\right)^{1/a}\right].$$

c) the derivative of joint cdf with respect to x is

$$\frac{\partial F(x,y)}{\partial x} = \exp(-x) - x^{a-1}\left(x^a + y^a\right)^{1/a-1} \exp\left[-\left(x^a + y^a\right)^{1/a}\right],$$

so the conditional cdf if Y given $X = x$ is

$$F(y|x) = 1 - x^{a-1}\left(x^a + y^a\right)^{1/a-1} \exp\left[x - \left(x^a + y^a\right)^{1/a}\right].$$

d) the derivative of joint cdf with respect to y is

$$\frac{\partial F(x,y)}{\partial y} = \exp(-y) - y^{a-1}\left(x^a + y^a\right)^{1/a-1} \exp\left[-\left(x^a + y^a\right)^{1/a}\right],$$

so the conditional cdf if X given $Y = y$ is

$$F(x|y) = 1 - y^{a-1}\left(x^a + y^a\right)^{1/a-1} \exp\left[y - \left(x^a + y^a\right)^{1/a}\right].$$

e) the derivative of joint cdf with respect to x and y is

$$\begin{aligned} f(x,y) &= \frac{\partial F(x,y)}{\partial x \partial y} \\ &= (xy)^{a-1}\left(x^a + y^a\right)^{1/a-2} \exp\left[-\left(x^a + y^a\right)^{1/a}\right] \\ &\quad \cdot \left[a - 1 + \left(x^a + y^a\right)^{1/a}\right]. \end{aligned}$$

Solutions to Question 2

a) Let X denote the actual stock return. The pdf of X is

$$\begin{aligned}
f_X(x) &= \frac{1}{b-a}\int_a^b \lambda \exp(-\lambda x) d\lambda \\
&= \frac{1}{b-a}\left\{\left[\lambda \frac{\exp(-\lambda x)}{-x}\right]_a^b + \frac{1}{x}\int_a^b \exp(-\lambda x) d\lambda\right\} \\
&= \frac{1}{b-a}\left\{-\frac{b\exp(-bx) - a\exp(-ax)}{x} - \frac{\exp(-bx) - \exp(-ax)}{x^2}\right\} \\
&= \frac{(xa+1)\exp(-ax) - (xb+1)\exp(-bx)}{x^2(b-a)}.
\end{aligned}$$

b) the expected value of X is

$$\begin{aligned}
E(X) &= \int_0^\infty \frac{(xa+1)\exp(-ax) - (xb+1)\exp(-bx)}{x(b-a)} dx \\
&= \frac{1}{b-a}\left[a\int_0^\infty \exp(-ax)dx - b\int_0^\infty \exp(-bx)dx + \int_0^\infty \frac{1}{x}\exp(-ax)dx - \int_0^\infty \frac{1}{x}\exp(-bx)dx\right] \\
&= \frac{1}{b-a}\left[1 - 1 + \int_0^\infty \frac{1}{x}\exp(-ax)dx - \int_0^\infty \frac{1}{x}\exp(-bx)dx\right] \\
&= \frac{1}{b-a}\left[\int_0^\infty \frac{1}{x}\exp(-ax)dx - \int_0^\infty \frac{1}{x}\exp(-bx)dx\right] \\
&= \frac{1}{b-a}\left[\infty - \infty\right] \\
&= \infty.
\end{aligned}$$

c) the expected value of X^2 is

$$\begin{aligned}
E\left(X^2\right) &= \int_0^\infty \frac{(xa+1)\exp(-ax) - (xb+1)\exp(-bx)}{b-a} dx \\
&= \frac{1}{b-a}\left[a\int_0^\infty x\exp(-ax)dx - b\int_0^\infty x\exp(-bx)dx + \int_0^\infty \exp(-ax)dx - \int_0^\infty \exp(-bx)dx\right] \\
&= \frac{1}{b-a}\left[\frac{1}{a} - \frac{1}{b} + \frac{1}{a} - \frac{1}{b}\right] \\
&= \frac{2}{ab}.
\end{aligned}$$

Hence, the variance is infinite.

d) If $x_1, x_2, \ldots, x_n$ is a random sample on X then the likelihood function is

$$L(a,b) = (b-a)^{-n}\prod_{i=1}^n \frac{(x_i a + 1)\exp\left(-a x_i\right) - (x_i b + 1)\exp\left(-b x_i\right)}{x_i^2}.$$

The log-likelihood function is

$$\log L(a, b) = -n \log(b - a) + \sum_{i=1}^{n} \log \left[(x_i a + 1) \exp(-a x_i) - (x_i b + 1) \exp(-b x_i) \right] - 2 \sum_{i=1}^{n} \log x_i.$$

The partial derivatives with respect to a and b are

$$\frac{\partial \log L}{\partial a} = \frac{n}{b - a} - a \sum_{i=1}^{n} \frac{x_i^2 \exp(-a x_i)}{(x_i a + 1) \exp(-a x_i) - (x_i b + 1) \exp(-b x_i)}$$

and

$$\frac{\partial \log L}{\partial b} = -\frac{n}{b - a} - b \sum_{i=1}^{n} \frac{x_i^2 \exp(-b x_i)}{(x_i a + 1) \exp(-a x_i) - (x_i b + 1) \exp(-b x_i)}.$$

So, the mles of a and b are the simultaneous solutions of the equations

$$\frac{n}{b - a} = a \sum_{i=1}^{n} \frac{x_i^2 \exp(-a x_i)}{(x_i a + 1) \exp(-a x_i) - (x_i b + 1) \exp(-b x_i)}$$

and

$$-\frac{n}{b - a} = b \sum_{i=1}^{n} \frac{x_i^2 \exp(-b x_i)}{(x_i a + 1) \exp(-a x_i) - (x_i b + 1) \exp(-b x_i)}.$$

Solutions to Question 3

If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cdf of a normalized version of M_n converges to G, i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n\left(a_n x + b_n\right) \rightarrow G(x) \tag{1}$$

as $n \rightarrow \infty$ then G must be of the same type as (cdfs G and G^* are of the same type if $G^*(x) = G(ax+b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$\begin{aligned}
I &: \Lambda(x) = \exp\left\{-\exp(-x)\right\}, \qquad x \in \Re; \\
II &: \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\left\{-x^{-\alpha}\right\} & \text{if } x \geq 0 \end{cases} \\
& \quad \text{for some } \alpha > 0; \\
III &: \Psi_\alpha(x) = \begin{cases} \exp\left\{-(-x)^{\alpha}\right\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \\
& \quad \text{for some } \alpha > 0.
\end{aligned}$$

The necessary and sufficient conditions for the three extreme value distributions are:

$$\begin{aligned}
I &: \exists \gamma(t) > 0 \text{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F\left(t + x\gamma(t)\right)}{1 - F(t)} = \exp(-x), \qquad x \in \Re, \\
II &: w(F) = \infty \text{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \qquad x > 0, \\
III &: w(F) < \infty \text{ and } \lim_{t \downarrow 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} = x^{\alpha}, \qquad x > 0.
\end{aligned}$$

Firstly, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function, say $h(t)$, such that

$$\lim_{t \rightarrow w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} = \exp(-x)$$

for every $x \in (-\infty, \infty)$. But, using L'Hopital's rule, we note that

$$\begin{aligned}
\lim_{t \rightarrow w(F)} \frac{1 - F\left(t + x\,h(t)\right)}{1 - F(t)} &= \lim_{t \rightarrow w(F)} \frac{[1 + x\,h^{'}(t)] f\left(t + x\,h(t)\right)}{f(t)} \\
&= \lim_{t \rightarrow w(G)} \frac{[1 + x\,h^{'}(t)] g(t + x\,h(t))}{g(t)} \left[\frac{G(t + x\,h(t))}{G(t)}\right]^{a-1} \\
& \quad \times \left[\frac{1 - G\left(t + x\,h(t)\right)}{1 - G(t)}\right]^{b-1} \exp\left\{c\,G(t) - c\,G\left(t + x\,h(t)\right)\right\} \\
&= \exp(-b\,x)
\end{aligned}$$

for every $x \in (-\infty, \infty)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n\to\infty} \Pr\left\{a_n\left(M_n - b_n\right) \le x\right\} = \exp\left\{-\exp(-b\,x)\right\}$$

for some suitable norming constants $a_n > 0$ and b_n.

Secondly, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta < 0$ such that

$$\lim_{t\to\infty} \frac{1 - G(t\,x)}{1 - G(t)} = x^{\beta}$$

for every $x > 0$. But, using L'Hopital's rule, we note that

$$\begin{aligned}
\lim_{t\to\infty} \frac{1 - F(t\,x)}{1 - F(t)} &= \lim_{t\to\infty} \frac{x f(t\,x)}{f(t)} \\
&= \lim_{t\to\infty} \frac{x\,g(t\,x)}{g(t)} \left[\frac{G(t\,x)}{G(t)}\right]^{a-1} \left[\frac{1 - G(t\,x)}{1 - G(t)}\right]^{b-1} \exp\left\{c\,G(t) - c\,G\left(t\,x\right)\right\} \\
&= \lim_{t\to\infty} \frac{x\,g(t\,x)}{g(t)} \left[\frac{1 - G(t\,x)}{1 - G(t)}\right]^{b-1} \\
&= \lim_{t\to\infty} \frac{1 - G(t\,x)}{1 - G(t)} \left[\frac{1 - G(t\,x)}{1 - G(t)}\right]^{b-1} \\
&= \lim_{t\to\infty} \left[\frac{1 - G(t\,x)}{1 - G(t)}\right]^{b} \\
&= x^{b\,\beta}
\end{aligned}$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n\to\infty} \Pr\left\{a_n\left(M_n - b_n\right) \le x\right\} = \exp\left(-x^{b\beta}\right)$$

for some suitable norming constants $a_n > 0$ and b_n.

Thirdly, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\alpha > 0$ such that

$$\lim_{t\to 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} = x^{\alpha}$$

for every $x > 0$. But, using L'Hopital's rule, we note that

$$\begin{aligned}
\lim_{t\to 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} &= \lim_{t\to 0} \frac{xf(w(F) - tx)}{f(w(F) - t)} \\
&= \lim_{t\to 0} \frac{xg(w(F) - tx)}{g(w(F) - t)} \left[\frac{G(w(F) - tx)}{G(w(F) - t)}\right]^{a-1} \left[\frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)}\right]^{b-1} \\
&\quad \times \exp\left\{cG(w(F) - t) - cG\left(w(F) - tx\right)\right\} \\
&= \lim_{t\to 0} \frac{xg(w(F) - tx)}{g(w(F) - t)} \left[\frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)}\right]^{b-1} \\
&= \lim_{t\to 0} \frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \left[\frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)}\right]^{b-1} \\
&= \lim_{t\to 0} \left[\frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)}\right]^{b} \\
&= x^{b\alpha}.
\end{aligned}$$

So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{n\to\infty} \Pr\left\{a_n\left(M_n - b_n\right) \leq x\right\} = \exp\left\{-(-x)^{b\alpha}\right\}$$

for some suitable norming constants $a_n > 0$ and b_n.

Solutions to Question 4

a) Note that $w(F) = \infty$. Then

$$\begin{aligned}
\lim_{t\uparrow\infty} \frac{1 - F\left(t + xg(t)\right)}{1 - F(t)} &= \lim_{t\uparrow\infty} \frac{1 - \left\{1 - \exp\left[1 - (1 + \lambda t + \lambda x g(t))^\alpha\right]\right\}}{1 - \left\{1 - \exp\left[1 - (1 + \lambda t)^\alpha\right]\right\}} \\
&= \lim_{t\uparrow\infty} \frac{\exp\left[1 - (1 + \lambda t + \lambda x g(t))^\alpha\right]}{\exp\left[1 - (1 + \lambda t)^\alpha\right]} \\
&= \lim_{t\uparrow\infty} \exp\left[(1 + \lambda t)^\alpha - (1 + \lambda t + \lambda x g(t))^\alpha\right] \\
&= \lim_{t\uparrow\infty} \exp\left\{(1 + \lambda t)^\alpha \left[1 - \left(1 + \frac{\lambda g(t) x}{1 + \lambda t}\right)^\alpha\right]\right\} \\
&= \lim_{t\uparrow\infty} \exp\left\{(1 + \lambda t)^\alpha \left[1 - \left(1 + \alpha \frac{\lambda g(t) x}{1 + \lambda t}\right)\right]\right\} \qquad \text{using } (1+x)^a \approx 1 + ax \\
&= \lim_{t\uparrow\infty} \exp\left\{-(1 + \lambda t)^\alpha \left[\alpha \frac{\lambda g(t) x}{1 + \lambda t}\right]\right\} \\
&= \lim_{t\uparrow\infty} \exp\left\{-\lambda\alpha (1 + \lambda t)^{\alpha - 1} g(t) x\right\} \\
&= \exp\left\{-x\right\}
\end{aligned}$$

if $g(t) = 1/(\lambda\alpha)(1+\lambda t)^{1-\alpha}$. So, the exponentiated extension cdf $F(x) = 1 - \exp\left[1 - (1 + \lambda x)^\alpha\right]^\alpha$ belongs to the Gumbel domain of attraction.

b) Note that $w(F) = \infty$. Then

$$\begin{aligned}
\lim_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} &= \lim_{t\to\infty} \frac{\left[1 - \exp\left(-\frac{\lambda}{tx}\right)\right]^\alpha}{\left[1 - \exp\left(-\frac{\lambda}{t}\right)\right]^\alpha} \\
&= \lim_{t\to\infty} \left[\frac{1 - \exp\left(-\frac{\lambda}{tx}\right)}{1 - \exp\left(-\frac{\lambda}{t}\right)}\right]^\alpha \\
&= \lim_{t\to\infty} \left[\frac{1 - \left(1 - \frac{\lambda}{tx}\right)}{1 - \left(1 - \frac{\lambda}{t}\right)}\right]^\alpha \qquad \text{using } \exp(-a) \approx 1 - a \\
&= \lim_{t\to\infty} \left[\frac{\frac{\lambda}{tx}}{\frac{\lambda}{t}}\right]^\alpha \\
&= x^{-\alpha}.
\end{aligned}$$

So, the inverse exponentiated exponential cdf $F(x) = 1 - \left[1 - \exp\left(-\frac{\lambda}{x}\right)\right]^\alpha$ belongs to the Fréchet domain of attraction.

c) For the Poisson distribution,

$$\frac{\Pr(X = k)}{1 - F(k - 1)} = \frac{\lambda^k / k!}{\sum_{j=k}^{\infty} \lambda^j / j!} = \frac{1}{1 + \sum_{j=k+1}^{\infty} k! \lambda^{j-k} / j!}.$$

The term in the denominator can be rewritten as

$$\sum_{j=1}^{\infty} \frac{\lambda^j}{(k+1)(k+2)\cdots(k+j)} \le \sum_{j=1}^{\infty} \left(\frac{\lambda}{k}\right)^j = \frac{\lambda/k}{1-\lambda/k}$$

(when $k > \lambda$) and the bound tends to 0 as $k \to \infty$ and so it follows that $p(k)/(1-F(k-1)) \to 1$. Hence, there can be no non-degenerate limit.

d) For the Bernoulli (p) distribution,

$$\frac{\Pr(X=k)}{1-F(k-1)} = \begin{cases} 1-p, & \text{if } k=0, \\ 1, & \text{if } k=1. \end{cases}$$

Hence, there can be no sequences $a_n > 0$ and b_n such that $(M_n - b_n)/a_n$ has a non-degenerate limiting distribution.

e) For the discrete Weibull distribution, the corresponding pmf is

$$p(x) = q^{x^a} - q^{(x+1)^a}.$$

So,

$$\begin{aligned} \frac{\Pr(X=x)}{1-F(x-1)} &= \frac{q^{x^a} - q^{(x+1)^a}}{1-\left[1-q^{x^a}\right]} \\ &= \frac{q^{x^a} - q^{(x+1)^a}}{q^{x^a}} \\ &= 1 - q^{(x+1)^a - x^a}. \end{aligned}$$

Note that

$$\begin{aligned} x^a - (x+1)^a &= x^a - x^a\left(1+\frac{1}{x}\right)^a \\ &= x^a\left[1-\left(1+\frac{1}{x}\right)^a\right] \\ &= x^a\left[1-1-a\frac{1}{x}-\frac{a(a-1)}{2!}\frac{1}{x^2}-\cdots\right] \\ &\to -\infty. \end{aligned}$$

Hence,

$$\frac{\Pr(X=x)}{1-F(x-1)} \to 1.$$

Hence, there can be no sequences $a_n > 0$ and b_n such that $(M_n - b_n)/a_n$ has a non-degenerate limiting distribution.

Solutions to Question 5

If X is an absolutely continuous random variable with cdf $F(\cdot)$ then

$$\mathrm{VaR}_p(X) = F^{-1}(p)$$

and

$$\mathrm{ES}_p(X) = \frac{1}{p} \int_0^p F^{-1}(v) dv.$$

Setting

$$\Phi\left(\frac{x-\mu}{\sigma}\right) = p$$

gives

$$\mathrm{VaR}_p(X) = \mu + \sigma \Phi^{-1}(p)$$

and

$$\mathrm{ES}_p(X) = \mu + \frac{\sigma}{p} \int_0^p \Phi^{-1}(v) dv.$$

a) The joint likelihood function of μ and σ^2 is

$$\begin{aligned} L\left(\mu, \sigma^2\right) &= \prod_{i=1}^n \left\{ \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{\left(X_i - \mu\right)^2}{2\sigma^2} \right] \right\} \\ &= \frac{1}{(2\pi)^{n/2}\sigma^n} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^n \left(X_i - \mu\right)^2 \right]. \end{aligned}$$

The joint log likelihood function of μ and σ^2 is

$$\log L\left(\mu, \sigma^2\right) = -\frac{n}{2} \log(2\pi) - n \log \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^n \left(X_i - \mu\right)^2.$$

The first order partial derivatives of this with respect to μ and σ are

$$\frac{\partial \log L}{\partial \mu} = \frac{1}{\sigma^2} \sum_{i=1}^n \left(X_i - \mu\right) = \frac{1}{\sigma^2} \left(\sum_{i=1}^n X_i - n\mu \right) \tag{2}$$

and

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^n \left(X_i - \mu\right)^2, \tag{3}$$

respectively.

b) Using equation (2), one can see that the solution of $\partial \log L / \partial \mu = 0$ is $\mu = \overline{X} = (1/n) \sum_{i=1}^{n} X_i$.

c) Using equation (3), one can see that the solution of $\partial \log L / \partial \sigma = 0$ is $\sigma^2 = S^2 = (1/n) \sum_{i=1}^{n} (X_i - \overline{X})^2$.

d) The mle of Value at Risk is

$$\widehat{\mathrm{VaR}}_p(X) = \overline{X} + S \Phi^{-1}(p)$$

The mle of Expected Shortfall is

$$\widehat{\mathrm{ES}}_p(X) = \overline{X} + \frac{S}{p} \int_0^p \Phi^{-1}(v) dv.$$

e) Since

$$\begin{aligned} E\left(\overline{X}\right) &= E\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) \\ &= \frac{1}{n} \sum_{i=1}^{n} E\left(X_i\right) \\ &= \frac{1}{n} \sum_{i=1}^{n} \mu \\ &= \mu, \end{aligned}$$

$\overline{X}$ is unbiased for μ. Since $\sum_{i=1}^{n} (X_i - \overline{X})^2 \sim \sigma^2 \chi_{n-1}^2$ and $E\left(\chi_k\right) = \sqrt{2} \Gamma\left((k+1)/2\right) / \Gamma(k/2)$, we can write

$$\begin{aligned} E(S) &= E\left[\frac{\sigma}{\sqrt{n}} \sqrt{\chi_{n-1}^2}\right] \\ &= \frac{\sigma}{\sqrt{n}} E\left[\sqrt{\chi_{n-1}^2}\right] \\ &= \frac{\sigma}{\sqrt{n}} \frac{\sqrt{2} \Gamma\left(n/2\right)}{\Gamma\left((n-1)/2\right)}, \end{aligned}$$

so S is biased for σ.

Since

$$\begin{aligned} E\left(\widehat{\mathrm{VaR}}_p(X)\right) &= E\left(\overline{X}\right) + E(S) \Phi^{-1}(p) \\ &= \mu + \frac{\sigma}{\sqrt{n}} \frac{\sqrt{2} \Gamma\left(n/2\right)}{\Gamma\left((n-1)/2\right)} \Phi^{-1}(p) \\ &\neq \mu + \sigma \Phi^{-1}(p), \end{aligned}$$

$\widehat{\mathrm{VaR}}_p(X)$ is biased for $\mathrm{VaR}_p(X)$.

f) Since

$$\begin{aligned} E\left(\widehat{\mathrm{ES}}_p(X)\right) &= E\left(\overline{X}\right) + E(S)\frac{1}{p}\int_0^p \Phi^{-1}(v)dv \\ &= \mu + \frac{\sigma}{\sqrt{n}}\frac{\sqrt{2}\Gamma\left(n/2\right)}{\Gamma\left((n-1)/2\right)}\frac{1}{p}\int_0^p \Phi^{-1}(v)dv \\ &\neq \mu + \sigma\frac{1}{p}\int_0^p \Phi^{-1}(v)dv, \end{aligned}$$

$\widehat{\mathrm{ES}}_p(X)$ is biased for $\mathrm{ES}_p(X)$.

Solutions to Question 6

a) The cdf of X is

$$\begin{aligned}
F_Y(y) &= \Pr(Y \leq y) \\
&= \Pr\left(\min\left(X_1, \ldots, X_\alpha\right) \leq y\right) \\
&= 1 - \Pr\left(\min\left(X_1, \ldots, X_\alpha\right) > y\right) \\
&= 1 - \Pr\left(X_1 > y, \ldots, X_\alpha > y\right) \\
&= 1 - \Pr\left(X_1 > y\right) \cdots \Pr\left(X_\alpha > y\right) \\
&= 1 - \exp(-\lambda y) \cdots \exp(-\lambda y) \\
&= 1 - \exp(-\alpha \lambda y),
\end{aligned}$$

the exponential cdf with parameter $\alpha\lambda$.

b) The corresponding pdf is

$$f_Y(y) = \alpha\lambda \exp(-\alpha\lambda y).$$

c) The nth moment of Y can be calculated as

$$\begin{aligned}
E\left(Y^n\right) &= \alpha\lambda \int_0^\infty x^n \exp(-\alpha\lambda x) dx \\
&= (\alpha\lambda)^{-n} \int_0^\infty x^n \exp(-x) dx \\
&= (\alpha\lambda)^{-n} \Gamma(n+1) \\
&= (\alpha\lambda)^{-n} n!.
\end{aligned}$$

So,

$$E\left(Y\right) = (\alpha\lambda)^{-1}$$

and

$$Var\left(Y\right) = (\alpha\lambda)^{-2}.$$

d) Setting

$$1 - \exp(-\alpha\lambda y) = p$$

gives

$$\mathrm{VaR}_p(Y) = -\frac{1}{\alpha\lambda} \log\left(1 - p\right).$$

e) The expected shortfall is

$$\begin{aligned}
\mathrm{ES}_p(Y) &= -\frac{1}{\alpha\lambda p}\int_0^p \log(1-v)\,dv \\
&= -\frac{1}{\alpha\lambda p}\left\{[v\log(1-v)]_0^p + \int_0^p \frac{v}{1-v}dv\right\} \\
&= -\frac{1}{\alpha\lambda p}\left\{p\log(1-p) + \int_0^p \frac{v-1+1}{1-v}dv\right\} \\
&= -\frac{1}{\alpha\lambda p}\left\{p\log(1-p) - p + \int_0^p \frac{1}{1-v}dv\right\} \\
&= -\frac{1}{\alpha\lambda p}\left\{p\log(1-p) - p - \log(1-p)\right\}.
\end{aligned}$$

f) The likelihood function is

$$L(\alpha,\lambda) = \alpha^n\lambda^n \exp\left(-\alpha\lambda\sum_{i=1}^n y_i\right).$$

The log-likelihood function is

$$\log L = n\log(\alpha\lambda) - \alpha\lambda\sum_{i=1}^n y_i.$$

The partial derivatives with respect to α and λ are

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} - \lambda\sum_{i=1}^n y_i$$

and

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \alpha\sum_{i=1}^n y_i.$$

Setting these to zero, we find that the mles of α and λ are the solutions of

$$\widehat{\alpha} = \frac{n}{\lambda\sum_{i=1}^n y_i}.$$

By definition, α must be a positive integer. Hence, the set of all possible mles of α and λ is $\left\{\left(m, \frac{n}{m\sum_{i=1}^n y_i}\right), m = 1, 2, \ldots\right\}$.