SOLUTIONS TO MATH68181 EXTREME VALUES EXAM

Solutions to Question 1 a) We can write

$$\overline{F}(x,y) = \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right] = \exp\left\{-(x+y)\left[\theta\frac{y^2}{(x+y)^2} - \theta\frac{y}{x+y} + 1\right]\right\}.$$

This is in the form of

$$\overline{F}(x,y) = \exp\left[-(x+y)A\left(\frac{y}{x+y}\right)\right]$$

with $A(t) = \theta t^2 - \theta t + 1$.

We now check the conditions for $A(\cdot)$. Clearly, $A(0) = 1$ and $A(1) = 1$.

Also $A(t) \geq 0$ since

$$\begin{aligned} & \theta t^2 - \theta t + 1 \geq 0 \\ \Leftrightarrow \quad & \theta\left(t^2 - t\right) + 1 \geq 0 \\ \Leftrightarrow \quad & \theta(t - 1/2)^2 + 1 - \theta/4 \geq 0, \end{aligned}$$

which always holds.

Also $A(t) \leq 1$ since

$$\begin{aligned} & \theta t^2 - \theta t + 1 \leq 1 \\ \Leftrightarrow \quad & \theta t^2 - \theta t \leq 0 \\ \Leftrightarrow \quad & \theta t(t-1) \leq 0, \end{aligned}$$

which always holds.

Also $A(t) \geq t$ since

$$\begin{aligned} & \theta t^2 - \theta t + 1 \geq t \\ \Leftrightarrow \quad & \theta t^2 - (\theta + 1)t + 1 \geq 0 \\ \Leftrightarrow \quad & (1 - \theta t)(1 - t) \geq 0, \end{aligned}$$

which always holds.

Also $A(t) \geq 1 - t$ since

$$\begin{aligned} & \theta t^2 - \theta t + 1 \geq 1 - t \\ \Leftrightarrow \quad & \theta t^2 + (1 - \theta)t \geq 0 \\ \Leftrightarrow \quad & (\theta t - \theta + 1)t \geq 0, \end{aligned}$$

which always holds.

$A^{'}(t) = 2\theta t - \theta$ and $A^{''}(t) = 2\theta > 0$, so $A(\cdot)$ is convex.

b) the joint cdf is

$$F(x,y) = 1 - \exp(-x) - \exp(-y) + \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right].$$

c) the derivative of joint cdf with respect to x is

$$\frac{\partial F(x,y)}{\partial x} = \exp(-x) + \left[\frac{\theta y^2}{(x+y)^2} - 1\right] \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right],$$

so the conditional cdf if Y given $X = x$ is

$$F(y|x) = 1 + \left[\frac{\theta y^2}{(x+y)^2} - 1\right] \exp\left[-\frac{\theta y^2}{x+y} + \theta y - y\right].$$

d) the derivative of joint cdf with respect to y is

$$\frac{\partial F(x,y)}{\partial y} = \exp(-y) + \left[\frac{\theta y^2}{(x+y)^2} - \frac{2\theta y}{x+y} + \theta - 1\right] \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right],$$

so the conditional cdf if X given $Y = y$ is

$$F(x|y) = 1 + \left[\frac{\theta y^2}{(x+y)^2} - \frac{2\theta y}{x+y} + \theta - 1\right] \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x\right].$$

e) the derivative of joint cdf with respect to x and y is

$$\begin{aligned} f(x,y) &= \frac{\partial F(x,y)}{\partial x \partial y} \\ &= \left\{\left[\frac{\theta y^2}{(x+y)^2} - 1\right]\left[\frac{\theta y^2}{(x+y)^2} - \frac{2\theta y}{x+y} + \theta - 1\right] + \left[\frac{2\theta y}{(x+y)^2} - \frac{2\theta y^2}{(x+y)^3}\right]\right\} \\ &\quad \times \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right]. \end{aligned}$$

Solutions to Question 2 a) Let X denote the actual stock return. The pdf of X is

$$\begin{aligned} f_X(x) &= \int_0^\infty \lambda \exp(-\lambda x) a \exp(-a\lambda) d\lambda \\ &= a \int_0^\infty \lambda \exp\left\{-\lambda(x+a)\right\} d\lambda \\ &= \frac{a}{(x+a)^2}. \end{aligned}$$

b) the expected value of X is

$$\begin{aligned} E(X) &= \int_0^\infty \frac{ax}{(x+a)^2} dx \\ &= a \int_0^1 \frac{1-y}{y} dy \\ &= a \int_0^1 \frac{1}{y} dy - a \\ &= a \left[-\log y\right]_0^1 - a \\ &= \infty. \end{aligned}$$

b) the expected value of X^2 is

$$\begin{aligned} E\left(X^2\right) &= \int_0^\infty \frac{ax^2}{(x+a)^2} dx \\ &= a^2 \int_0^1 \frac{(1-y)^2}{y^2} dy \\ &> a^2 \int_0^1 \frac{(1-y)^2}{y} dy \\ &= a^2 \int_0^1 \frac{1}{y} dy - 2a^2 \int_0^1 dy + a^2 \int_0^1 y dy \\ &= a^2 \left[\log y\right]_0^1 - 2a^2 \left[y\right]_0^1 + a^2 \left[y^2/2\right]_0^1 \\ &= a^2 \left[0 - (-\infty)\right] - 2a^2 \left[1 - 0\right] + a^2 \left[1/2 - 0\right] \\ &= \infty. \end{aligned}$$

Hence, the variance is infinite.

d) If $x_1, x_2, \ldots, x_n$ is a random sample on X then the likelihood function is

$$L(a) = a^n \prod_{i=1}^n (x_i + a)^{-2}.$$

The log-likelihood function is

$$\log L = n \log a - 2 \sum_{i=1}^n \log(x_i + a).$$

The derivative with respect to a is

$$\frac{d\log L}{da} = \frac{n}{a} - 2\sum_{i=1}^{n} \frac{1}{x_i + a}.$$

So, the mle of a is the root of the equation

$$\frac{n}{a} = 2\sum_{i=1}^{n} \frac{1}{x_i + a}.$$

Solutions to Question 3 If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cdf of a normalized version of M_n converges to G, i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n\left(a_n x + b_n\right) \to G(x) \tag{1}$$

as $n \to \infty$ then G must be of the same type as (cdfs G and G^* are of the same type if $G^*(x) = G(ax + b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$\begin{aligned}
I &: \Lambda(x) = \exp\left\{-\exp(-x)\right\}, \qquad x \in \Re; \\
II &: \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\left\{-x^{-\alpha}\right\} & \text{if } x \geq 0 \end{cases} \\
&\quad \text{for some } \alpha > 0; \\
III &: \Psi_\alpha(x) = \begin{cases} \exp\left\{-(-x)^{\alpha}\right\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \\
&\quad \text{for some } \alpha > 0.
\end{aligned}$$

The necessary and sufficient conditions for the three extreme value distributions are:

$$\begin{aligned}
I &: \exists \gamma(t) > 0 \text{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F\left(t + x\gamma(t)\right)}{1 - F(t)} = \exp(-x), \qquad x \in \Re, \\
II &: w(F) = \infty \text{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \qquad x > 0, \\
III &: w(F) < \infty \text{ and } \lim_{t \downarrow 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} = x^{\alpha}, \qquad x > 0.
\end{aligned}$$

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function, say $h(t)$, such that

$$\lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} = \exp(-x)$$

for every $x \in (-\infty, \infty)$. But, using L'Hopital's rule, we note that

$$
\begin{aligned}
& \lim_{t \to w(F)} \frac{1 - F\left(t + xh(t)\right)}{1 - F(t)} \\
= & \lim_{t \to w(F)} \frac{\left(1 + xh^{'}(t)\right) f\left(t + xh(t)\right)}{f(t)} \\
= & \lim_{t \to w(G)} \frac{\left(1 + xh^{'}(t)\right) \left\{-\log\left[1 - G\left(t + xh(t)\right)\right]\right\}^{a-1} g\left(t + xh(t)\right)}{\left\{-\log\left[1 - G(t)\right]\right\}^{a-1} g(t)} \\
= & \lim_{t \to w(G)} \frac{\left(1 + xh^{'}(t)\right) g\left(t + xh(t)\right)}{g(t)} \left\{\frac{\log\left[1 - G\left(t + xh(t)\right)\right]}{\log\left[1 - G(t)\right]}\right\}^{a-1} \\
= & \lim_{t \to w(G)} \frac{\left(1 + xh^{'}(t)\right) g\left(t + xh(t)\right)}{g(t)} \left\{\frac{1 - G\left(t\right)}{1 - G\left(t + xh(t)\right)} \frac{\left(1 + xh^{'}(t)\right) g\left(t + xh(t)\right)}{g(t)}\right\}^{a-1} \\
= & \lim_{t \to w(G)} \frac{\left(1 + xh^{'}(t)\right) g\left(t + xh(t)\right)}{g(t)} \left\{\frac{g\left(t\right)}{\left(1 + xh^{'}(t)\right) g\left(t + xh(t)\right)} \frac{\left(1 + xh^{'}(t)\right) g\left(t + xh(t)\right)}{g(t)}\right\}^{a-1} \\
= & \lim_{t \to w(G)} \frac{\left(1 + xh^{'}(t)\right) g\left(t + xh(t)\right)}{g(t)} \\
= & \lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} \\
= & \exp(-x)
\end{aligned}
$$

for every $x \in (-\infty, \infty)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$
\lim_{n \to \infty} \Pr\left\{a_n\left(M_n - b_n\right) \leq x\right\} = \exp\left\{-\exp(-x)\right\}
$$

for some suitable norming constants $a_n > 0$ and b_n.

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$, such that

$$
\lim_{t \to \infty} \frac{1 - G(tx)}{1 - G(t)} = x^{\beta}
$$

for every $x > 0$. But, using L'Hopital's rule, we note that

$$\begin{aligned}
\lim_{t\to\infty}\frac{1-F(tx)}{1-F(t)} &= \lim_{t\to\infty}\frac{xf(tx)}{f(t)} \\
&= \lim_{t\to\infty}\frac{x\left\{-\log\left[1-G(tx)\right]\right\}^{a-1}g(tx)}{\left\{-\log\left[1-G(t)\right]\right\}^{a-1}g(t)} \\
&= \lim_{t\to\infty}\frac{xg(tx)}{g(t)}\left\{\frac{\log\left[1-G(tx)\right]}{\log\left[1-G(t)\right]}\right\}^{a-1} \\
&= \lim_{t\to\infty}\frac{xg(tx)}{g(t)}\left\{\frac{1-G(t)}{1-G(tx)}\frac{xg(tx)}{g(t)}\right\}^{a-1} \\
&= \lim_{t\to\infty}\frac{xg(tx)}{g(t)}\left\{\frac{g(t)}{xg(tx)}\frac{xg(tx)}{g(t)}\right\}^{a-1} \\
&= \lim_{t\to\infty}\frac{xg(tx)}{g(t)} \\
&= \lim_{t\to\infty}\frac{1-G(tx)}{1-G(t)} \\
&= x^{\beta}
\end{aligned}$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n\to\infty}\Pr\left\{a_n\left(M_n-b_n\right)\le x\right\} = \exp\left(-x^{\beta}\right)$$

for some suitable norming constants $a_n > 0$ and b_n.

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\alpha > 0$, such that

$$\lim_{t\to 0}\frac{1-G(w(G)-tx)}{1-G(w(G)-t)} = x^{\alpha}$$

for every $x > 0$. But, using L'Hopital's rule, we note that

$$\begin{aligned}
\lim_{t \to 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} &= \lim_{t \to 0} \frac{x f(w(F) - tx)}{f(w(F) - t)} \\
&= \lim_{t \to 0} \frac{x \left\{ -\log \left[1 - G(w(F) - tx) \right] \right\}^{a-1} g(w(F) - tx)}{\left\{ -\log \left[1 - G(w(F) - tx) \right] \right\}^{a-1} g(w(F) - t)} \\
&= \lim_{t \to 0} \frac{x g(w(F) - tx)}{g(w(F) - t)} \left\{ \frac{\log \left[1 - G(w(F) - tx) \right]}{\log \left[1 - G(w(F) - t) \right]} \right\}^{a-1} \\
&= \lim_{t \to 0} \frac{x g(w(F) - tx)}{g(w(F) - t)} \left\{ \frac{1 - G(w(F) - t)}{1 - G(w(F) - tx)} \frac{x g(w(F) - tx)}{g(w(F) - t)} \right\}^{a-1} \\
&= \lim_{t \to 0} \frac{x g(w(F) - tx)}{g(w(F) - t)} \left\{ \frac{g(w(F) - t)}{x g(w(F) - tx)} \frac{x g(w(F) - tx)}{g(w(F) - t)} \right\}^{a-1} \\
&= \lim_{t \to 0} \frac{x g(w(F) - tx)}{g(w(F) - t)} \\
&= \lim_{t \to 0} \frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \\
&= x^{\alpha}.
\end{aligned}$$

So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{n \to \infty} \Pr \left\{ a_n \left(M_n - b_n \right) \leq x \right\} = \exp \left\{ -(-x)^{\alpha} \right\}$$

for some suitable norming constants $a_n > 0$ and b_n.

Solutions to Question 4 i) Note that $w(F)=\infty$ and take $\gamma(t)=1$. Then

$$\lim_{t\uparrow\infty}\frac{1-F\left(t+x\right)}{1-F(t)}=\lim_{t\uparrow\infty}\frac{1-\left[1-\exp\left(-t-x\right)\right]^{\alpha}}{1-\left[1-\exp\left(-t\right)\right]^{\alpha}}=\lim_{t\uparrow\infty}\frac{\alpha\exp\left(-t-x\right)}{\alpha\exp\left(-t\right)}=\exp(-x).$$

. So, the exponentiated exponential cdf $F(x)=[1-\exp(-x)]^{\alpha}$ belongs to the Gumbel domain of attraction.

ii) Note that $w(F)=\infty$ and take $\gamma(t)=1/\beta$. Then

$$\begin{aligned}\lim_{t\to\infty}\frac{1-F(t+x/\beta)}{1-F(t)} &= \lim_{t\to\infty}\frac{1-\{1-\exp(-\beta t-x)\}^{\alpha}}{1-p+p\left\{1-\exp(-\beta t-x)\right\}^{\alpha}}\frac{1-p+p\left\{1-\exp(-\beta t)\right\}^{\alpha}}{1-\{1-\exp(-\beta t)\}^{\alpha}}\\ &= \lim_{t\to\infty}\frac{\alpha\exp(-\beta t-x)}{\alpha\exp(-\beta t)}\\ &= \exp(-x).\end{aligned}$$

So, the exponentiated exponential geometric cdf $F(x)=\frac{\{1-\exp(-\beta x)\}^{\alpha}}{1-p+p\{1-\exp(-\beta x)\}^{\alpha}}$ belongs to the Gumbel domain of attraction.

iii) Note that $w(F)=\infty$. Then

$$\begin{aligned}\lim_{t\to\infty}\frac{1-F(t+x\gamma(t))}{1-F(t)} &= \frac{1-\left\{1-\dfrac{(1-p)\exp\left(-k\beta\left(t+x\gamma(t)\right)\right)}{\left[1-p\exp\left(-\beta\left(t+x\gamma(t)\right)\right)\right]^{k}}\right\}}{1-\left\{1-\dfrac{(1-p)\exp\left(-k\beta\left(t\right)\right)}{\left[1-p\exp\left(-\beta\left(t\right)\right)\right]^{k}}\right\}}\\ &= \frac{\dfrac{(1-p)\exp\left(-k\beta\left(t+x\gamma(t)\right)\right)}{\left[1-p\exp\left(-\beta\left(t+x\gamma(t)\right)\right)\right]^{k}}}{\dfrac{(1-p)\exp\left(-k\beta\left(t\right)\right)}{\left[1-p\exp\left(-\beta\left(t\right)\right)\right]^{k}}}\\ &= \frac{\dfrac{\exp\left(-k\beta\left(t+x\gamma(t)\right)\right)}{\left[1-p\exp\left(-\beta\left(t+x\gamma(t)\right)\right)\right]^{k}}}{\dfrac{\exp\left(-k\beta\left(t\right)\right)}{\left[1-p\exp\left(-\beta\left(t\right)\right)\right]^{k}}}\\ &= \frac{\dfrac{\exp\left(-k\beta\left(t+x\gamma(t)\right)\right)}{\left[1-0\right]^{k}}}{\dfrac{\exp\left(-k\beta\left(t\right)\right)}{\left[1-0\right]^{k}}}\\ &= \frac{\exp\left(-k\beta\left(t+x\gamma(t)\right)\right)}{\exp\left(-k\beta\left(t\right)\right)}\\ &= \exp\left(-k\beta x\gamma(t)\right)\\ &= \exp\left(-x\right)\end{aligned}$$

if $\gamma(t)=1/(k\beta)$. So, the exponential-negative binomial distribution cdf $F(x)=1-\frac{(1-p)^{k}\exp(-k\beta x)}{[1-p\exp(-\beta x)]^{k}}$ belongs to the Gumbel domain of attraction.

iv) For the degenerate distribution,

$$p(k) = \begin{cases} 1, & \text{if } k = k_0, \\ 0, & \text{if } k \neq k_0. \end{cases}$$

So,

$$F(k) = \begin{cases} 1, & \text{if } k \geq k_0, \\ 0, & \text{if } k < k_0, \end{cases}$$

and

$$\frac{\Pr(X = k)}{1 - F(k-1)} = \begin{cases} 1/1, & \text{if } k = k_0, \\ 0/1, & \text{if } k_0 < k < k_0 + 1, \\ 0/0, & \text{if } k \geq k_0 + 1, \\ 0/1, & \text{if } k < k_0. \end{cases}$$

Hence, there can be no sequences $a_n > 0$ and b_n such that $(M_n - b_n)/a_n$ has a non-degenerate limiting distribution.

v) For the Poisson distribution,

$$\begin{aligned} \frac{\Pr(X = k)}{1 - F(k-1)} &= \frac{\lambda^k/k!}{\sum_{j=k}^{\infty} \lambda^j/j!} \\ &= \frac{1}{1 + \sum_{j=k+1}^{\infty} k!\lambda^{j-k}/j!}. \end{aligned}$$

The term in the denominator of the last term can be rewritten as

$$\sum_{j=1}^{\infty} \frac{\lambda^j}{(k+1)(k+2)\cdots(k+j)} \leq \sum_{j=1}^{\infty} \left(\frac{\lambda}{k}\right)^j = \frac{\lambda/k}{1 - \lambda/k}$$

(when $k > \lambda$) and the bound tends to 0 as $k \to \infty$ and so it follows that $p_k/(1 - F(k-1)) \to 1$. Hence, there can be no sequences $a_n > 0$ and b_n such that $(M_n - b_n)/a_n$ has a non-degenerate limiting distribution.

Solutions to Question 5 If X is an absolutely continuous random variable with cdf $F(\cdot)$ then

$$\mathrm{VaR}_p(X) = F^{-1}(p)$$

and

$$\mathrm{ES}_p(X) = \frac{1}{p}\int_0^p F^{-1}(v)dv.$$

Setting

$$\frac{x-a}{b-a} = p$$

gives

$$\mathrm{VaR}_p(X) = a + p(b-a).$$

So,

$$\begin{aligned}
\mathrm{ES}_p(X) &= \frac{1}{p}\int_0^p \left[a + v(b-a)\right] dv \\
&= \frac{1}{p}\left[ap + \frac{p^2}{2}(b-a)\right] \\
&= a + \frac{p}{2}(b-a).
\end{aligned}$$

i) The joint likelihood function of a and b is

$$\begin{aligned}
L\left(a,b\right) &= \frac{1}{b-a}I\{a < X_1 < b\}\frac{1}{b-a}I\{a < X_2 < b\}\cdots\frac{1}{b-a}I\{a < X_n < b\} \\
&= \frac{1}{(b-a)^n}\prod_{i=1}^{n} I\{a < X_i < b\} \\
&= \frac{1}{(b-a)^n}I\{\max\left(X_1, X_2, \ldots, X_n\right) < b\}I\{a < \min\left(X_1, X_2, \ldots, X_n\right)\},
\end{aligned}$$

where $I\{\cdot\}$ denotes the indicator function.

ii) Note that $(b-a)^{-n}$ is an increasing function of a over $(-\infty, \min(X_1, X_2, \ldots, X_n))$. So, the maximum of $L(a,b)$ will be attained at $a = \min(X_1, X_2, \ldots, X_n)$.

iii) Note that $(b-a)^{-n}$ is a decreasing function of b over $(\max(X_1, X_2, \ldots, X_n), \infty)$. So, the maximum of $L(a,b)$ will be attained at $b = \max(X_1, X_2, \ldots, X_n)$.

iv) The mle of Value at Risk is

$$\widehat{\mathrm{VaR}}_p(X) = \min\left(X_1, X_2, \ldots, X_n\right) + p\left[\max\left(X_1, X_2, \ldots, X_n\right) - \min\left(X_1, X_2, \ldots, X_n\right)\right].$$

The mle of Expected Shortfall is

$$\widehat{\mathrm{ES}}_p(X) = \min\left(X_1, X_2, \ldots, X_n\right) + \frac{p}{2}\left[\max\left(X_1, X_2, \ldots, X_n\right) - \min\left(X_1, X_2, \ldots, X_n\right)\right].$$

v) Let $Z = \min(X_1, X_2, \ldots, X_n)$. Then

$$F_Z(z) = 1 - \left(\frac{b-z}{b-a}\right)^n$$

and

$$f_Z(z) = n\frac{(b-z)^{n-1}}{(b-a)^n}$$

for $a < z < b$, so

$$\begin{aligned}
E(Z) &= n\int_a^b z\frac{(b-z)^{n-1}}{(b-a)^n}dz \\
&= n\int_a^b (b-(b-z))\frac{(b-z)^{n-1}}{(b-a)^n}dz \\
&= nb\int_a^b \frac{(b-z)^{n-1}}{(b-a)^n}dz - n\int_a^b \frac{(b-z)^n}{(b-a)^n}dz \\
&= b + \frac{n(b-a)}{n+1}.
\end{aligned}$$

vi) Now let $Z = \max(X_1, X_2, \ldots, X_n)$. Then

$$F_Z(z) = \left(\frac{z-a}{b-a}\right)^n$$

and

$$f_Z(z) = n\frac{(z-a)^{n-1}}{(b-a)^n}$$

for $a < z < b$, so

$$\begin{aligned}
E(Z) &= n\int_a^b z\frac{(z-a)^{n-1}}{(b-a)^n}dz \\
&= n\int_a^b (z-a+a)\frac{(z-a)^{n-1}}{(b-a)^n}dz \\
&= n\int_a^b \frac{(z-a)^n}{(b-a)^n}dz + na\int_a^b \frac{(z-a)^{n-1}}{(b-a)^n}dz \\
&= \frac{n(b-a)}{n+1} + a \\
&= \frac{a}{n+1} + \frac{nb}{n+1},
\end{aligned}$$

Hence, $\widehat{\mathrm{VaR}}_p(X)$ and $\widehat{\mathrm{ES}}_p(X)$ are biased.

Solutions to Question 6 a) The cdf of $M = \max(X, Y, Z)$ is

$$\begin{aligned}
F_M(m) &= \Pr(X < m, Y < m, Z < m) \\
&= 1 - \Pr(X > m) - \Pr(Y > m) - \Pr(Z > m) + \Pr(X > m, Y > m) \\
&\quad + \Pr(X > m, Z > m) + \Pr(Y > m, Z > m) - \Pr(X > m, Y > m, Z > m) \\
&= 1 - \left[1 + \frac{m}{a}\right]^{-d} - \left[1 + \frac{m}{b}\right]^{-d} - \left[1 + \frac{m}{c}\right]^{-d} + \left[1 + \frac{m}{a} + \frac{m}{b}\right]^{-d} \\
&\quad + \left[1 + \frac{m}{a} + \frac{m}{c}\right]^{-d} + \left[1 + \frac{m}{b} + \frac{m}{c}\right]^{-d} - \left[1 + \frac{m}{a} + \frac{m}{b} + \frac{m}{c}\right]^{-d}.
\end{aligned}$$

b) Differentiating $F_M(m)$ with respect to m gives the pdf of M as

$$\begin{aligned}
f_M(m) &= \frac{d}{a}\left[1 + \frac{m}{a}\right]^{-d-1} + \frac{d}{b}\left[1 + \frac{m}{b}\right]^{-d-1} + \frac{d}{c}\left[1 + \frac{m}{c}\right]^{-d-1} - d\left(\frac{1}{a} + \frac{1}{b}\right)\left[1 + \frac{m}{a} + \frac{m}{b}\right]^{-d-1} \\
&\quad - d\left(\frac{1}{a} + \frac{1}{c}\right)\left[1 + \frac{m}{a} + \frac{m}{c}\right]^{-d-1} - d\left(\frac{1}{b} + \frac{1}{c}\right)\left[1 + \frac{m}{b} + \frac{m}{c}\right]^{-d-1} \\
&\quad + d\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left[1 + \frac{m}{a} + \frac{m}{b} + \frac{m}{c}\right]^{-d-1}.
\end{aligned}$$

c) The nth moment of M can be calculated as

$$\begin{aligned}
E\left(M^n\right) &= \frac{d}{a}\int_0^\infty m^n\left[1 + \frac{m}{a}\right]^{-d-1} dm \\
&\quad + \frac{d}{b}\int_0^\infty m^n\left[1 + \frac{m}{b}\right]^{-d-1} dm \\
&\quad + \frac{d}{c}\int_0^\infty m^n\left[1 + \frac{m}{c}\right]^{-d-1} dm \\
&\quad - d\left(\frac{1}{a} + \frac{1}{b}\right)\int_0^\infty m^n\left[1 + \frac{m}{a} + \frac{m}{b}\right]^{-d-1} dm \\
&\quad - d\left(\frac{1}{a} + \frac{1}{c}\right)\int_0^\infty m^n\left[1 + \frac{m}{a} + \frac{m}{c}\right]^{-d-1} dm \\
&\quad - d\left(\frac{1}{b} + \frac{1}{c}\right)\int_0^\infty m^n\left[1 + \frac{m}{b} + \frac{m}{c}\right]^{-d-1} dm \\
&\quad + d\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\int_0^\infty m^n\left[1 + \frac{m}{a} + \frac{m}{b} + \frac{m}{c}\right]^{-d-1} dm
\end{aligned}$$

$$= \frac{d}{a}\int_0^\infty m^n \left[1+\frac{m}{a}\right]^{-d-1}\frac{dm}{dx}dx$$

$$\left[\text{set } x=\left[1+\frac{m}{a}\right]^{-1},\ \text{so, } m=a\frac{1-x}{x} \text{ and } \frac{dm}{dx}=-\frac{a}{x^2}\right]$$

$$+\frac{d}{b}\int_0^\infty m^n \left[1+\frac{m}{b}\right]^{-d-1}\frac{dm}{dx}dx$$

$$\left[\text{set } x=\left[1+\frac{m}{b}\right]^{-1},\ \text{so, } m=b\frac{1-x}{x} \text{ and } \frac{dm}{dx}=-\frac{b}{x^2}\right]$$

$$+\frac{d}{c}\int_0^\infty m^n \left[1+\frac{m}{c}\right]^{-d-1}\frac{dm}{dx}dx$$

$$\left[\text{set } x=\left[1+\frac{m}{c}\right]^{-1},\ \text{so, } m=c\frac{1-x}{x} \text{ and } \frac{dm}{dx}=-\frac{c}{x^2}\right]$$

$$-d\left(\frac{1}{a}+\frac{1}{b}\right)\int_0^\infty m^n \left[1+\frac{m}{a}+\frac{m}{b}\right]^{-d-1}\frac{dm}{dx}dx$$

$$\left[\text{set } x=\left[1+\frac{m}{a}+\frac{m}{b}\right]^{-1},\ \text{so, } m=\left(\frac{1}{a}+\frac{1}{b}\right)^{-1}\frac{1-x}{x} \text{ and } \frac{dm}{dx}=-\left(\frac{1}{a}+\frac{1}{b}\right)^{-1}\frac{1}{x^2}\right]$$

$$-d\left(\frac{1}{a}+\frac{1}{c}\right)\int_0^\infty m^n \left[1+\frac{m}{a}+\frac{m}{c}\right]^{-d-1}\frac{dm}{dx}dx$$

$$\left[\text{set } x=\left[1+\frac{m}{a}+\frac{m}{c}\right]^{-1},\ \text{so, } m=\left(\frac{1}{a}+\frac{1}{c}\right)^{-1}\frac{1-x}{x} \text{ and } \frac{dm}{dx}=-\left(\frac{1}{a}+\frac{1}{c}\right)^{-1}\frac{1}{x^2}\right]$$

$$-d\left(\frac{1}{a}+\frac{1}{c}\right)\int_0^\infty m^n \left[1+\frac{m}{b}+\frac{m}{c}\right]^{-d-1}\frac{dm}{dx}dx$$

$$\left[\text{set } x=\left[1+\frac{m}{b}+\frac{m}{c}\right]^{-1},\ \text{so, } m=\left(\frac{1}{b}+\frac{1}{c}\right)^{-1}\frac{1-x}{x} \text{ and } \frac{dm}{dx}=-\left(\frac{1}{b}+\frac{1}{c}\right)^{-1}\frac{1}{x^2}\right]$$

$$+d\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\int_0^\infty m^n \left[1+\frac{m}{a}+\frac{m}{b}+\frac{m}{c}\right]^{-d-1}\frac{dm}{dx}dx$$

$$\left[\text{set } x=\left[1+\frac{m}{a}+\frac{m}{b}+\frac{m}{c}\right]^{-1},\ \text{so, } m=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{-1}\frac{1-x}{x} \text{ and } \frac{dm}{dx}=-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{-1}\frac{1}{x^2}\right]$$

$$
\begin{aligned}
= \quad & -\frac{d}{a}\int_1^0 a^n \frac{(1-x)^n}{x^n} x^{d+1} \frac{a}{x^2} dx \\
& -\frac{d}{b}\int_1^0 b^n \frac{(1-x)^n}{x^n} x^{d+1} \frac{b}{x^2} dx \\
& -\frac{d}{c}\int_1^0 c^n \frac{(1-x)^n}{x^n} x^{d+1} \frac{c}{x^2} dx \\
& +d\left(\frac{1}{a}+\frac{1}{b}\right)\int_1^0 \left(\frac{1}{a}+\frac{1}{b}\right)^{-n} \frac{(1-x)^n}{x^n} x^{d+1} \left(\frac{1}{a}+\frac{1}{b}\right)^{-1} \frac{1}{x^2} dx \\
& +d\left(\frac{1}{a}+\frac{1}{c}\right)\int_1^0 \left(\frac{1}{a}+\frac{1}{c}\right)^{-n} \frac{(1-x)^n}{x^n} x^{d+1} \left(\frac{1}{a}+\frac{1}{c}\right)^{-1} \frac{1}{x^2} dx \\
& +d\left(\frac{1}{b}+\frac{1}{c}\right)\int_1^0 \left(\frac{1}{b}+\frac{1}{c}\right)^{-n} \frac{(1-x)^n}{x^n} x^{d+1} \left(\frac{1}{b}+\frac{1}{c}\right)^{-1} \frac{1}{x^2} dx \\
& -d\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\int_1^0 \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{-n} \frac{(1-x)^n}{x^n} x^{d+1} \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{-1} \frac{1}{x^2} dx \\
= \quad & \frac{d}{a}a^{n+1}\int_0^1 x^{d-n-1}(1-x)^n dx + \frac{d}{b}b^{n+1}\int_0^1 x^{d-n-1}(1-x)^n dx \\
& +\frac{d}{c}c^{n+1}\int_0^1 x^{d-n-1}(1-x)^n dx - d\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{a}+\frac{1}{b}\right)^{-n-1}\int_0^1 x^{d-n-1}(1-x)^n dx \\
& -d\left(\frac{1}{a}+\frac{1}{c}\right)\left(\frac{1}{a}+\frac{1}{c}\right)^{-n-1}\int_0^1 x^{d-n-1}(1-x)^n dx \\
& -d\left(\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{b}+\frac{1}{c}\right)^{-n-1}\int_0^1 x^{d-n-1}(1-x)^n dx \\
& +d\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{-n-1}\int_0^1 x^{d-n-1}(1-x)^n dx \\
= \quad & da^n B(d-n,n+1) + db^n B(d-n,n+1) \\
& +dc^n B(d-n,n+1) - d\left(\frac{1}{a}+\frac{1}{b}\right)^{-n} B(d-n,n+1) \\
& -d\left(\frac{1}{a}+\frac{1}{c}\right)^{-n} B(d-n,n+1) \\
& -d\left(\frac{1}{b}+\frac{1}{c}\right)^{-n} B(d-n,n+1) \\
& +d\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{-n} B(d-n,n+1).
\end{aligned}
$$

d) The cdf of $L = \min(X, Y, Z)$ is

$$\begin{aligned} F_L(l) &= 1 - \Pr(L > l) \\ &= 1 - \Pr(X > l, Y > l, Z > l) \\ &= 1 - \left[1 + \frac{l}{a} + \frac{l}{b} + \frac{l}{c}\right]^{-d}. \end{aligned}$$

e) Differentiating $F_L(l)$ with respect to l gives the pdf of L as

$$f_L(l) = d\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left[1 + \frac{l}{a} + \frac{l}{b} + \frac{l}{c}\right]^{-d-1}.$$

f) The nth moment of L can be calculated as

$$\begin{aligned} E\left(L^n\right) &= d\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\int_0^\infty l^n \left[1 + \frac{l}{a} + \frac{l}{b} + \frac{l}{c}\right]^{-d-1} dl \\ &= d\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^{-n} B(d-n, n+1). \end{aligned}$$

Solutions to Question 7 (i) The cdf of X is

$$\begin{aligned}
F_Y(y) &= \Pr(Y \leq y) \\
&= \Pr\left(\max\left(X_1, \ldots, X_\alpha\right) \leq y\right) \\
&= \Pr\left(X_1 \leq x, \ldots, X_\alpha \leq y\right) \\
&= \Pr\left(X_1 \leq y\right) \cdots \Pr\left(X_\alpha \leq y\right) \\
&= [1 - \exp(-\lambda y)] \cdots [1 - \exp(-\lambda y)] \\
&= [1 - \exp(-\lambda y)]^\alpha .
\end{aligned}$$

(ii) Differentiating

$$F_Y(y) = [1 - \exp(-\lambda y)]^\alpha$$

with respect to y gives the pdf of Y as

$$f_Y(y) = \alpha\lambda \exp(-\lambda y) \left[1 - \exp(-\lambda y)\right]^{\alpha - 1} .$$

(iii) The nth moment of Y can be calculated as

$$\begin{aligned}
E\left(Y^n\right) &= \alpha\lambda \int_0^\infty x^n \exp(-\lambda x) \left[1 - \exp(-\lambda x)\right]^{\alpha-1} dx \\
&= (-1)^n \alpha \lambda^{-n} \int_0^1 (\log y)^n \left[1 - y\right]^{\alpha - 1} dy \\
&= (-1)^n \alpha \lambda^{-n} \left.\frac{\partial^n}{\partial \beta^n} \int_0^1 y^\beta \left[1 - y\right]^{\alpha - 1} dy \right|_{\beta = 0} \\
&= (-1)^n \alpha \lambda^{-n} \left.\frac{\partial^n}{\partial \beta^n} B(\beta + 1, \alpha) \right|_{\beta = 0} .
\end{aligned}$$

(iv) Setting

$$[1 - \exp(-\lambda y)]^\alpha = p$$

gives

$$\mathrm{VaR}_p(Y) = -\frac{1}{\lambda} \log\left(1 - p^{1/\alpha}\right) .$$

(v) The expected shortfall is

$$\begin{aligned}
\mathrm{ES}_p(Y) &= -\frac{1}{\lambda p} \int_0^p \log\left(1 - v^{1/\alpha}\right) dv \\
&= \frac{1}{\lambda p} \int_0^p \sum_{i=0}^\infty \frac{v^{i/\alpha}}{i} dv \\
&= \frac{1}{\lambda p} \sum_{i=0}^\infty \int_0^p \frac{v^{i/\alpha}}{i} dv \\
&= \frac{1}{\lambda p} \sum_{i=0}^\infty \frac{p^{i/\alpha + 1}}{i\left(i/\alpha + 1\right)} .
\end{aligned}$$

(vi) The likelihood function is

$$L(\alpha, \lambda) = \alpha^n \lambda^n \exp\left(-\lambda \sum_{i=1}^{n} y_i\right) \left\{\prod_{i=1}^{n} \left[1 - \exp\left(-\lambda y_i\right)\right]\right\}^{\alpha-1}.$$

The log-likelihood function is

$$\log L = n \log(\alpha\lambda) - \lambda \sum_{i=1}^{n} y_i + (\alpha - 1) \sum_{i=1}^{n} \log\left[1 - \exp\left(-\lambda y_i\right)\right].$$

The partial derivatives with respect to α and λ are

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left[1 - \exp\left(-\lambda y_i\right)\right]$$

and

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} y_i + (\alpha - 1) \sum_{i=1}^{n} \frac{y_i \exp\left(-\lambda y_i\right)}{1 - \exp\left(-\lambda y_i\right)}.$$

Setting these to zero, we obtain the mle of α as

$$\widehat{\alpha} = -\frac{n}{\displaystyle\sum_{i=1}^{n} \log\left[1 - \exp\left(-\lambda y_i\right)\right]}.$$

The mle of λ is the root of the equation

$$\frac{n}{\lambda} - \sum_{i=1}^{n} y_i - \left(\frac{n}{\displaystyle\sum_{i=1}^{n} \log\left[1 - \exp\left(-\lambda y_i\right)\right]} + 1\right) \sum_{i=1}^{n} \frac{y_i \exp\left(-\lambda y_i\right)}{1 - \exp\left(-\lambda y_i\right)}.$$

Solutions to Question 8 a) If $X_i \sim Exp(\lambda_i)$, $i = 1, 2$ are independent random variables then

$$\begin{aligned}
\Pr(X_1 < X_2) &= \int_0^\infty \left[1 - \exp(-\lambda_1 x_2)\right] \lambda_2 \exp(-\lambda_2 x_2)\, dx_2 \\
&= 1 - \int_0^\infty \lambda_2 \exp(-\lambda_1 x_2 - \lambda_2 x_2)\, dx_2 \\
&= 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}.
\end{aligned}$$

b) If $X_i \sim Exp(\lambda_i)$, $i = 1, 2, 3$ are independent random variables then

$$\begin{aligned}
\Pr(X_1 < X_2 < X_3) &= \int_0^\infty \int_{x_1}^\infty \int_{x_2}^\infty \lambda_1 \lambda_2 \lambda_3 \exp(-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_3)\, dx_3 dx_2 dx_1 \\
&= \int_0^\infty \int_{x_1}^\infty \lambda_1 \lambda_2 \exp(-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_2)\, dx_2 dx_1 \\
&= \frac{\lambda_1 \lambda_2}{\lambda_2 + \lambda_3} \int_0^\infty \exp(-\lambda_1 x_1 - \lambda_2 x_1 - \lambda_3 x_1)\, dx_1 \\
&= \frac{\lambda_1 \lambda_2}{(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3)}.
\end{aligned}$$

c) If $X_i \sim Exp(\lambda_i)$, $i = 1, 2, 3, 4$ are independent random variables then

$$\begin{aligned}
& \Pr(X_1 < X_2 < X_3 < X_4) \\
&= \int_0^\infty \int_{x_1}^\infty \int_{x_2}^\infty \int_{x_3}^\infty \lambda_1 \lambda_2 \lambda_3 \lambda_4 \exp(-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_3 - \lambda_4 x_4)\, dx_4 dx_3 dx_2 dx_1 \\
&= \int_0^\infty \int_{x_1}^\infty \int_{x_2}^\infty \lambda_1 \lambda_2 \lambda_3 \exp(-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_3 - \lambda_4 x_3)\, dx_3 dx_2 dx_1 \\
&= \frac{\lambda_1 \lambda_2 \lambda_3}{(\lambda_3 + \lambda_4)} \int_0^\infty \int_{x_1}^\infty \exp(-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_2 - \lambda_4 x_2)\, dx_2 dx_1 \\
&= \frac{\lambda_1 \lambda_2 \lambda_3}{(\lambda_3 + \lambda_4)(\lambda_2 + \lambda_3 + \lambda_4)} \int_0^\infty \exp(-\lambda_1 x_1 - \lambda_2 x_1 - \lambda_3 x_1 - \lambda_4 x_1)\, dx_2 dx_1 \\
&= \frac{\lambda_1 \lambda_2 \lambda_3}{(\lambda_3 + \lambda_4)(\lambda_2 + \lambda_3 + \lambda_4)(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)}.
\end{aligned}$$

If $X_i \sim Exp(\lambda_i)$, $i = 1, 2, \ldots, k$ are independent random variables then the general formula will be

$$\Pr(X_1 < X_2 < \cdots < X_k) = \frac{\lambda_1 \lambda_2 \cdots \lambda_k}{(\lambda_k + \lambda_{k-1})(\lambda_k + \lambda_{k-1} + \lambda_{k-2}) \cdots (\lambda_k + \lambda_{k-1} + \cdots + \lambda_1)}.$$

This can be proved by induction.

If $\lambda_i = \lambda$ then

$$\Pr\left(X_1 < X_2\right) = \frac{1}{2},$$

$$\Pr\left(X_1 < X_2 < X_3\right) = \frac{2}{2 \cdot 3},$$

$$\Pr\left(X_1 < X_2 < X_3 < X_4\right) = \frac{1}{2 \cdot 3 \cdot 4}$$

and

$$\Pr\left(X_1 < X_2 < \cdots < X_k\right) = \frac{1}{2 \cdot 3 \cdots k} = \frac{1}{k!}.$$