

**SOLUTIONS TO
MATH48181
EXTREME VALUES EXAM**

Solutions to Question 1 a) We can write

$$\bar{F}(x, y) = \exp \left[-\frac{\theta y^2}{x+y} + \theta y - x - y \right] = \exp \left\{ -(x+y) \left[\theta \frac{y^2}{(x+y)^2} - \theta \frac{y}{x+y} + 1 \right] \right\}.$$

This is in the form of

$$\bar{F}(x, y) = \exp \left[-(x+y) A \left(\frac{y}{x+y} \right) \right]$$

with $A(t) = \theta t^2 - \theta t + 1$.

We now check the conditions for $A(\cdot)$. Clearly, $A(0) = 1$ and $A(1) = 1$.

Also $A(t) \geq 0$ since

$$\begin{aligned} & \theta t^2 - \theta t + 1 \geq 0 \\ \Leftrightarrow & \theta (t^2 - t) + 1 \geq 0 \\ \Leftrightarrow & \theta (t - 1/2)^2 + 1 - \theta/4 \geq 0, \end{aligned}$$

which always holds.

Also $A(t) \leq 1$ since

$$\begin{aligned} & \theta t^2 - \theta t + 1 \leq 1 \\ \Leftrightarrow & \theta t^2 - \theta t \leq 0 \\ \Leftrightarrow & \theta t(t - 1) \leq 0, \end{aligned}$$

which always holds.

Also $A(t) \geq t$ since

$$\begin{aligned} & \theta t^2 - \theta t + 1 \geq t \\ \Leftrightarrow & \theta t^2 - (\theta + 1)t + 1 \geq 0 \\ \Leftrightarrow & (1 - \theta t)(1 - t) \geq 0, \end{aligned}$$

which always holds.

Also $A(t) \geq 1 - t$ since

$$\begin{aligned} & \theta t^2 - \theta t + 1 \geq 1 - t \\ \Leftrightarrow & \theta t^2 + (1 - \theta)t \geq 0 \\ \Leftrightarrow & (\theta t - \theta + 1)t \geq 0, \end{aligned}$$

which always holds.

$A'(t) = 2\theta t - \theta$ and $A''(t) = 2\theta > 0$, so $A(\cdot)$ is convex.

b) the joint cdf is

$$F(x, y) = 1 - \exp(-x) - \exp(-y) + \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right].$$

c) the derivative of joint cdf with respect to x is

$$\frac{\partial F(x, y)}{\partial x} = \exp(-x) + \left[\frac{\theta y^2}{(x+y)^2} - 1\right] \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right],$$

so the conditional cdf if Y given $X = x$ is

$$F(y|x) = 1 + \left[\frac{\theta y^2}{(x+y)^2} - 1\right] \exp\left[-\frac{\theta y^2}{x+y} + \theta y - y\right].$$

d) the derivative of joint cdf with respect to y is

$$\frac{\partial F(x, y)}{\partial y} = \exp(-y) + \left[\frac{\theta y^2}{(x+y)^2} - \frac{2\theta y}{x+y} + \theta - 1\right] \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right],$$

so the conditional cdf if X given $Y = y$ is

$$F(x|y) = 1 + \left[\frac{\theta y^2}{(x+y)^2} - \frac{2\theta y}{x+y} + \theta - 1\right] \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x\right].$$

e) the derivative of joint cdf with respect to x and y is

$$\begin{aligned} f(x, y) &= \frac{\partial F(x, y)}{\partial x \partial y} \\ &= \left\{ \left[\frac{\theta y^2}{(x+y)^2} - 1 \right] \left[\frac{\theta y^2}{(x+y)^2} - \frac{2\theta y}{x+y} + \theta - 1 \right] + \left[\frac{2\theta y}{(x+y)^2} - \frac{2\theta y^2}{(x+y)^3} \right] \right\} \\ &\quad \times \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right]. \end{aligned}$$

Solutions to Question 2 a) Let X denote the actual stock return. The pdf of X is

$$\begin{aligned} f_X(x) &= \int_0^\infty \lambda \exp(-\lambda x) a \exp(-a\lambda) d\lambda \\ &= a \int_0^\infty \lambda \exp\{-\lambda(x+a)\} d\lambda \\ &= \frac{a}{(x+a)^2}. \end{aligned}$$

b) the expected value of X is

$$\begin{aligned} E(X) &= \int_0^\infty \frac{ax}{(x+a)^2} dx \\ &= a \int_0^1 \frac{1-y}{y} dy \\ &= a \int_0^1 \frac{1}{y} dy - a \\ &= a [-\log y]_0^1 - a \\ &= \infty. \end{aligned}$$

b) the expected value of X^2 is

$$\begin{aligned} E(X^2) &= \int_0^\infty \frac{ax^2}{(x+a)^2} dx \\ &= a^2 \int_0^1 \frac{(1-y)^2}{y^2} dy \\ &> a^2 \int_0^1 \frac{(1-y)^2}{y} dy \\ &= a^2 \int_0^1 \frac{1}{y} dy - 2a^2 \int_0^1 dy + a^2 \int_0^1 y dy \\ &= a^2 [\log y]_0^1 - 2a^2 [y]_0^1 + a^2 [y^2/2]_0^1 \\ &= a^2 [0 - (-\infty)] - 2a^2 [1 - 0] + a^2 [1/2 - 0] \\ &= \infty. \end{aligned}$$

Hence, the variance is infinite.

d) If x_1, x_2, \dots, x_n is a random sample on X then the likelihood function is

$$L(a) = a^n \prod_{i=1}^n (x_i + a)^{-2}.$$

The log-likelihood function is

$$\log L = n \log a - 2 \sum_{i=1}^n \log(x_i + a).$$

The derivative with respect to a is

$$\frac{d \log L}{da} = \frac{n}{a} - 2 \sum_{i=1}^n \frac{1}{x_i + a}.$$

So, the mle of a is the root of the equation

$$\frac{n}{a} = 2 \sum_{i=1}^n \frac{1}{x_i + a}.$$

Solutions to Question 3 If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cdf of a normalized version of M_n converges to G , i.e.

$$\Pr \left(\frac{M_n - b_n}{a_n} \leq x \right) = F^n(a_n x + b_n) \rightarrow G(x) \quad (1)$$

as $n \rightarrow \infty$ then G must be of the same type as (cdf's G and G^* are of the same type if $G^*(x) = G(ax + b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$\begin{aligned} I & : \Lambda(x) = \exp \{ -\exp(-x) \}, \quad x \in \mathfrak{R}; \\ II & : \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp \{ -x^{-\alpha} \} & \text{if } x \geq 0 \end{cases} \\ & \quad \text{for some } \alpha > 0; \\ III & : \Psi_\alpha(x) = \begin{cases} \exp \{ -(-x)^\alpha \} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \\ & \quad \text{for some } \alpha > 0. \end{aligned}$$

The necessary and sufficient conditions for the three extreme value distributions are:

$$\begin{aligned} I & : \exists \gamma(t) > 0 \text{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = \exp(-x), \quad x \in \mathfrak{R}, \\ II & : w(F) = \infty \text{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \quad x > 0, \\ III & : w(F) < \infty \text{ and } \lim_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} = x^\alpha, \quad x > 0. \end{aligned}$$

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function, say $h(t)$, such that

$$\lim_{t \rightarrow w(G)} \frac{1 - G(t + xh(t))}{1 - G(t)} = \exp(-x)$$

for every $x \in (-\infty, \infty)$. But, using L'Hopital's rule, we note that

$$\begin{aligned}
& \lim_{t \rightarrow w(F)} \frac{1 - F(t + xh(t))}{1 - F(t)} \\
&= \lim_{t \rightarrow w(F)} \frac{(1 + xh'(t)) f(t + xh(t))}{f(t)} \\
&= \lim_{t \rightarrow w(G)} \frac{(1 + xh'(t)) \{-\log[1 - G(t + xh(t))]\}^{a-1} g(t + xh(t))}{\{-\log[1 - G(t)]\}^{a-1} g(t)} \\
&= \lim_{t \rightarrow w(G)} \frac{(1 + xh'(t)) g(t + xh(t))}{g(t)} \left\{ \frac{\log[1 - G(t + xh(t))]}{\log[1 - G(t)]} \right\}^{a-1} \\
&= \lim_{t \rightarrow w(G)} \frac{(1 + xh'(t)) g(t + xh(t))}{g(t)} \left\{ \frac{1 - G(t)}{1 - G(t + xh(t))} \frac{(1 + xh'(t)) g(t + xh(t))}{g(t)} \right\}^{a-1} \\
&= \lim_{t \rightarrow w(G)} \frac{(1 + xh'(t)) g(t + xh(t))}{g(t)} \left\{ \frac{g(t)}{(1 + xh'(t)) g(t + xh(t))} \frac{(1 + xh'(t)) g(t + xh(t))}{g(t)} \right\}^{a-1} \\
&= \lim_{t \rightarrow w(G)} \frac{(1 + xh'(t)) g(t + xh(t))}{g(t)} \\
&= \lim_{t \rightarrow w(G)} \frac{1 - G(t + xh(t))}{1 - G(t)} \\
&= \exp(-x)
\end{aligned}$$

for every $x \in (-\infty, \infty)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n \rightarrow \infty} \Pr \{a_n (M_n - b_n) \leq x\} = \exp \{-\exp(-x)\}$$

for some suitable norming constants $a_n > 0$ and b_n .

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$, such that

$$\lim_{t \rightarrow \infty} \frac{1 - G(tx)}{1 - G(t)} = x^\beta$$

for every $x > 0$. But, using L'Hopital's rule, we note that

$$\begin{aligned}
\lim_{t \rightarrow \infty} \frac{1 - F(tx)}{1 - F(t)} &= \lim_{t \rightarrow \infty} \frac{xf(tx)}{f(t)} \\
&= \lim_{t \rightarrow \infty} \frac{x \{-\log [1 - G(tx)]\}^{a-1} g(tx)}{\{-\log [1 - G(t)]\}^{a-1} g(t)} \\
&= \lim_{t \rightarrow \infty} \frac{xg(tx)}{g(t)} \left\{ \frac{\log [1 - G(tx)]}{\log [1 - G(t)]} \right\}^{a-1} \\
&= \lim_{t \rightarrow \infty} \frac{xg(tx)}{g(t)} \left\{ \frac{1 - G(t)}{1 - G(tx)} \frac{xg(tx)}{g(t)} \right\}^{a-1} \\
&= \lim_{t \rightarrow \infty} \frac{xg(tx)}{g(t)} \left\{ \frac{g(t)}{xg(tx)} \frac{xg(tx)}{g(t)} \right\}^{a-1} \\
&= \lim_{t \rightarrow \infty} \frac{xg(tx)}{g(t)} \\
&= \lim_{t \rightarrow \infty} \frac{1 - G(tx)}{1 - G(t)} \\
&= x^\beta
\end{aligned}$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n \rightarrow \infty} \Pr \{a_n (M_n - b_n) \leq x\} = \exp(-x^\beta)$$

for some suitable norming constants $a_n > 0$ and b_n .

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\alpha > 0$, such that

$$\lim_{t \rightarrow 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} = x^\alpha$$

for every $x > 0$. But, using L'Hopital's rule, we note that

$$\begin{aligned}
\lim_{t \rightarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} &= \lim_{t \rightarrow 0} \frac{xf(w(F) - tx)}{f(w(F) - t)} \\
&= \lim_{t \rightarrow 0} \frac{x \{-\log [1 - G(w(F) - tx)]\}^{a-1} g(w(F) - tx)}{\{-\log [1 - G(w(F) - t)]\}^{a-1} g(w(F) - t)} \\
&= \lim_{t \rightarrow 0} \frac{xg(w(F) - tx)}{g(w(F) - t)} \left\{ \frac{\log [1 - G(w(F) - tx)]}{\log [1 - G(w(F) - t)]} \right\}^{a-1} \\
&= \lim_{t \rightarrow 0} \frac{xg(w(F) - tx)}{g(w(F) - t)} \left\{ \frac{1 - G(w(F) - t)}{1 - G(w(F) - tx)} \frac{xg(w(F) - tx)}{g(w(F) - t)} \right\}^{a-1} \\
&= \lim_{t \rightarrow 0} \frac{xg(w(F) - tx)}{g(w(F) - t)} \left\{ \frac{g(w(F) - t)}{xg(w(F) - tx)} \frac{xg(w(F) - tx)}{g(w(F) - t)} \right\}^{a-1} \\
&= \lim_{t \rightarrow 0} \frac{xg(w(F) - tx)}{g(w(F) - t)} \\
&= \lim_{t \rightarrow 0} \frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \\
&= x^\alpha.
\end{aligned}$$

So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{n \rightarrow \infty} \Pr \{a_n (M_n - b_n) \leq x\} = \exp \{-(-x)^\alpha\}$$

for some suitable norming constants $a_n > 0$ and b_n .

Solutions to Question 4 i) Note that $w(F) = \infty$ and take $\gamma(t) = 1$. Then

$$\lim_{t \uparrow \infty} \frac{1 - F(t+x)}{1 - F(t)} = \lim_{t \uparrow \infty} \frac{1 - [1 - \exp(-t-x)]^\alpha}{1 - [1 - \exp(-t)]^\alpha} = \lim_{t \uparrow \infty} \frac{\alpha \exp(-t-x)}{\alpha \exp(-t)} = \exp(-x).$$

. So, the exponentiated exponential cdf $F(x) = [1 - \exp(-x)]^\alpha$ belongs to the Gumbel domain of attraction.

ii) Note that $w(F) = \infty$ and take $\gamma(t) = 1/\beta$. Then

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1 - F(t+x/\beta)}{1 - F(t)} &= \lim_{t \rightarrow \infty} \frac{1 - \{1 - \exp(-\beta t - x)\}^\alpha}{1 - p + p \{1 - \exp(-\beta t - x)\}^\alpha} \frac{1 - p + p \{1 - \exp(-\beta t)\}^\alpha}{1 - \{1 - \exp(-\beta t)\}^\alpha} \\ &= \lim_{t \rightarrow \infty} \frac{\alpha \exp(-\beta t - x)}{\alpha \exp(-\beta t)} \\ &= \exp(-x). \end{aligned}$$

So, the exponentiated exponential geometric cdf $F(x) = \frac{\{1 - \exp(-\beta x)\}^\alpha}{1 - p + p \{1 - \exp(-\beta x)\}^\alpha}$ belongs to the Gumbel domain of attraction.

iii) the correspond pdf is

$$f(x) = \frac{k\beta(1-p)^k \exp(-k\beta x)}{[1 - p \exp(-\beta x)]^{k+1}}.$$

Note that $w(F) = \infty$ and take $\gamma(t) = 1/(k\beta)$. Since $f(x) \sim k\beta(1-p)^k \exp(-k\beta x)$ as $x \rightarrow \infty$,

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1 - F(t+x\gamma(t))}{1 - F(t)} &= \lim_{t \rightarrow \infty} \frac{f(t+x/(k\beta))}{f(t)} \\ &= \exp(-x). \end{aligned}$$

So, the exponential-negative binomial distribution cdf $F(x) = 1 - \frac{(1-p)^k \exp(-k\beta x)}{[1 - p \exp(-\beta x)]^k}$ belongs to the Gumbel domain of attraction.

iv) For the degenerate distribution,

$$p(k) = \begin{cases} 1, & \text{if } k = k_0, \\ 0, & \text{if } k \neq k_0. \end{cases}$$

So,

$$F(k) = \begin{cases} 1, & \text{if } k \geq k_0, \\ 0, & \text{if } k < k_0, \end{cases}$$

and

$$\frac{\Pr(X = k)}{1 - F(k-1)} = \begin{cases} 1/1, & \text{if } k = k_0, \\ 0/1, & \text{if } k_0 < k < k_0 + 1, \\ 0/0, & \text{if } k \geq k_0 + 1, \\ 0/1, & \text{if } k < k_0. \end{cases}$$

Hence, there can be no sequences $a_n > 0$ and b_n such that $(M_n - b_n)/a_n$ has a non-degenerate limiting distribution.

v) For the Poisson distribution,

$$\begin{aligned} \frac{\Pr(X = k)}{1 - F(k - 1)} &= \frac{\lambda^k/k!}{\sum_{j=k}^{\infty} \lambda^j/j!} \\ &= \frac{1}{1 + \sum_{j=k+1}^{\infty} k! \lambda^{j-k}/j!}. \end{aligned}$$

The term in the denominator of the last term can be rewritten as

$$\sum_{j=1}^{\infty} \frac{\lambda^j}{(k+1)(k+2)\cdots(k+j)} \leq \sum_{j=1}^{\infty} \left(\frac{\lambda}{k}\right)^j = \frac{\lambda/k}{1 - \lambda/k}$$

(when $k > \lambda$) and the bound tends to 0 as $k \rightarrow \infty$ and so it follows that $p_k/(1 - F(k-1)) \rightarrow 1$. Hence, there can be no sequences $a_n > 0$ and b_n such that $(M_n - b_n)/a_n$ has a non-degenerate limiting distribution.

Solutions to Question 5 If X is an absolutely continuous random variable with cdf $F(\cdot)$ then

$$\text{VaR}_p(X) = F^{-1}(p)$$

and

$$\text{ES}_p(X) = \frac{1}{p} \int_0^p F^{-1}(v) dv.$$

Setting

$$\frac{x - a}{b - a} = p$$

gives

$$\text{VaR}_p(X) = a + p(b - a).$$

So,

$$\begin{aligned} \text{ES}_p(X) &= \frac{1}{p} \int_0^p [a + v(b - a)] dv \\ &= \frac{1}{p} \left[ap + \frac{p^2}{2}(b - a) \right] \\ &= a + \frac{p}{2}(b - a). \end{aligned}$$

i) The joint likelihood function of a and b is

$$\begin{aligned} L(a, b) &= \frac{1}{b - a} I\{a < X_1 < b\} \frac{1}{b - a} I\{a < X_2 < b\} \cdots \frac{1}{b - a} I\{a < X_n < b\} \\ &= \frac{1}{(b - a)^n} \prod_{i=1}^n I\{a < X_i < b\} \\ &= \frac{1}{(b - a)^n} I\{\max(X_1, X_2, \dots, X_n) < b\} I\{a < \min(X_1, X_2, \dots, X_n)\}, \end{aligned}$$

where $I\{\cdot\}$ denotes the indicator function.

ii) Note that $(b - a)^{-n}$ is an increasing function of a over $(-\infty, \min(X_1, X_2, \dots, X_n))$. So, the maximum of $L(a, b)$ will be attained at $a = \min(X_1, X_2, \dots, X_n)$.

iii) Note that $(b - a)^{-n}$ is a decreasing function of b over $(\max(X_1, X_2, \dots, X_n), \infty)$. So, the maximum of $L(a, b)$ will be attained at $b = \max(X_1, X_2, \dots, X_n)$.

iv) The mle of Value at Risk is

$$\widehat{\text{VaR}}_p(X) = \min(X_1, X_2, \dots, X_n) + p[\max(X_1, X_2, \dots, X_n) - \min(X_1, X_2, \dots, X_n)].$$

The mle of Expected Shortfall is

$$\widehat{\text{ES}}_p(X) = \min(X_1, X_2, \dots, X_n) + \frac{p}{2} [\max(X_1, X_2, \dots, X_n) - \min(X_1, X_2, \dots, X_n)].$$

v) Let $Z = \min(X_1, X_2, \dots, X_n)$. Then

$$F_Z(z) = 1 - \left(\frac{b-z}{b-a}\right)^n$$

and

$$f_Z(z) = n \frac{(b-z)^{n-1}}{(b-a)^n}$$

for $a < z < b$, so

$$\begin{aligned} E(Z) &= n \int_a^b z \frac{(b-z)^{n-1}}{(b-a)^n} dz \\ &= n \int_a^b (b - (b-z)) \frac{(b-z)^{n-1}}{(b-a)^n} dz \\ &= nb \int_a^b \frac{(b-z)^{n-1}}{(b-a)^n} dz - n \int_a^b \frac{(b-z)^n}{(b-a)^n} dz \\ &= b + \frac{n(b-a)}{n+1}. \end{aligned}$$

vi) Now let $Z = \max(X_1, X_2, \dots, X_n)$. Then

$$F_Z(z) = \left(\frac{z-a}{b-a}\right)^n$$

and

$$f_Z(z) = n \frac{(z-a)^{n-1}}{(b-a)^n}$$

for $a < z < b$, so

$$\begin{aligned} E(Z) &= n \int_a^b z \frac{(z-a)^{n-1}}{(b-a)^n} dz \\ &= n \int_a^b (z-a+a) \frac{(z-a)^{n-1}}{(b-a)^n} dz \\ &= n \int_a^b \frac{(z-a)^n}{(b-a)^n} dz + na \int_a^b \frac{(z-a)^{n-1}}{(b-a)^n} dz \\ &= \frac{n(b-a)}{n+1} + a \\ &= \frac{a}{n+1} + \frac{nb}{n+1}, \end{aligned}$$

Hence, $\widehat{\text{VaR}}_p(X)$ and $\widehat{\text{ES}}_p(X)$ are biased.

Solutions to Question 6 a) The cdf of $M = \max(X, Y, Z)$ is

$$\begin{aligned}
F_M(m) &= \Pr(X < m, Y < m, Z < m) \\
&= 1 - \Pr(X > m) - \Pr(Y > m) - \Pr(Z > m) + \Pr(X > m, Y > m) \\
&\quad + \Pr(X > m, Z > m) + \Pr(Y > m, Z > m) - \Pr(X > m, Y > m, Z > m) \\
&= 1 - \left[1 + \frac{m}{a}\right]^{-d} - \left[1 + \frac{m}{b}\right]^{-d} - \left[1 + \frac{m}{c}\right]^{-d} + \left[1 + \frac{m}{a} + \frac{m}{b}\right]^{-d} \\
&\quad + \left[1 + \frac{m}{a} + \frac{m}{c}\right]^{-d} + \left[1 + \frac{m}{b} + \frac{m}{c}\right]^{-d} - \left[1 + \frac{m}{a} + \frac{m}{b} + \frac{m}{c}\right]^{-d}.
\end{aligned}$$

b) Differentiating $F_M(m)$ with respect to m gives the pdf of M as

$$\begin{aligned}
f_M(m) &= \frac{d}{a} \left[1 + \frac{m}{a}\right]^{-d-1} + \frac{d}{b} \left[1 + \frac{m}{b}\right]^{-d-1} + \frac{d}{c} \left[1 + \frac{m}{c}\right]^{-d-1} - d \left(\frac{1}{a} + \frac{1}{b}\right) \left[1 + \frac{m}{a} + \frac{m}{b}\right]^{-d-1} \\
&\quad - d \left(\frac{1}{a} + \frac{1}{c}\right) \left[1 + \frac{m}{a} + \frac{m}{c}\right]^{-d-1} - d \left(\frac{1}{b} + \frac{1}{c}\right) \left[1 + \frac{m}{b} + \frac{m}{c}\right]^{-d-1} \\
&\quad + d \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \left[1 + \frac{m}{a} + \frac{m}{b} + \frac{m}{c}\right]^{-d-1}.
\end{aligned}$$

c) The n th moment of M can be calculated as

$$\begin{aligned}
E(M^n) &= \frac{d}{a} \int_0^\infty m^n \left[1 + \frac{m}{a}\right]^{-d-1} dm \\
&\quad + \frac{d}{b} \int_0^\infty m^n \left[1 + \frac{m}{b}\right]^{-d-1} dm \\
&\quad + \frac{d}{c} \int_0^\infty m^n \left[1 + \frac{m}{c}\right]^{-d-1} dm \\
&\quad - d \left(\frac{1}{a} + \frac{1}{b}\right) \int_0^\infty m^n \left[1 + \frac{m}{a} + \frac{m}{b}\right]^{-d-1} dm \\
&\quad - d \left(\frac{1}{a} + \frac{1}{c}\right) \int_0^\infty m^n \left[1 + \frac{m}{a} + \frac{m}{c}\right]^{-d-1} dm \\
&\quad - d \left(\frac{1}{b} + \frac{1}{c}\right) \int_0^\infty m^n \left[1 + \frac{m}{b} + \frac{m}{c}\right]^{-d-1} dm \\
&\quad + d \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \int_0^\infty m^n \left[1 + \frac{m}{a} + \frac{m}{b} + \frac{m}{c}\right]^{-d-1} dm
\end{aligned}$$

$$\begin{aligned}
&= \frac{d}{a} \int_0^\infty m^n \left[1 + \frac{m}{a}\right]^{-d-1} \frac{dm}{dx} dx \\
&\quad \left[\text{set } x = \left[1 + \frac{m}{a}\right]^{-1}, \text{ so, } m = a \frac{1-x}{x} \text{ and } \frac{dm}{dx} = -\frac{a}{x^2} \right] \\
&+ \frac{d}{b} \int_0^\infty m^n \left[1 + \frac{m}{b}\right]^{-d-1} \frac{dm}{dx} dx \\
&\quad \left[\text{set } x = \left[1 + \frac{m}{b}\right]^{-1}, \text{ so, } m = b \frac{1-x}{x} \text{ and } \frac{dm}{dx} = -\frac{b}{x^2} \right] \\
&+ \frac{d}{c} \int_0^\infty m^n \left[1 + \frac{m}{c}\right]^{-d-1} \frac{dm}{dx} dx \\
&\quad \left[\text{set } x = \left[1 + \frac{m}{c}\right]^{-1}, \text{ so, } m = c \frac{1-x}{x} \text{ and } \frac{dm}{dx} = -\frac{c}{x^2} \right] \\
&-d \left(\frac{1}{a} + \frac{1}{b}\right) \int_0^\infty m^n \left[1 + \frac{m}{a} + \frac{m}{b}\right]^{-d-1} \frac{dm}{dx} dx \\
&\quad \left[\text{set } x = \left[1 + \frac{m}{a} + \frac{m}{b}\right]^{-1}, \text{ so, } m = \left(\frac{1}{a} + \frac{1}{b}\right)^{-1} \frac{1-x}{x} \text{ and } \frac{dm}{dx} = -\left(\frac{1}{a} + \frac{1}{b}\right)^{-1} \frac{1}{x^2} \right] \\
&-d \left(\frac{1}{a} + \frac{1}{c}\right) \int_0^\infty m^n \left[1 + \frac{m}{a} + \frac{m}{c}\right]^{-d-1} \frac{dm}{dx} dx \\
&\quad \left[\text{set } x = \left[1 + \frac{m}{a} + \frac{m}{c}\right]^{-1}, \text{ so, } m = \left(\frac{1}{a} + \frac{1}{c}\right)^{-1} \frac{1-x}{x} \text{ and } \frac{dm}{dx} = -\left(\frac{1}{a} + \frac{1}{c}\right)^{-1} \frac{1}{x^2} \right] \\
&-d \left(\frac{1}{a} + \frac{1}{c}\right) \int_0^\infty m^n \left[1 + \frac{m}{b} + \frac{m}{c}\right]^{-d-1} \frac{dm}{dx} dx \\
&\quad \left[\text{set } x = \left[1 + \frac{m}{b} + \frac{m}{c}\right]^{-1}, \text{ so, } m = \left(\frac{1}{b} + \frac{1}{c}\right)^{-1} \frac{1-x}{x} \text{ and } \frac{dm}{dx} = -\left(\frac{1}{b} + \frac{1}{c}\right)^{-1} \frac{1}{x^2} \right] \\
&+d \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \int_0^\infty m^n \left[1 + \frac{m}{a} + \frac{m}{b} + \frac{m}{c}\right]^{-d-1} \frac{dm}{dx} dx \\
&\quad \left[\text{set } x = \left[1 + \frac{m}{a} + \frac{m}{b} + \frac{m}{c}\right]^{-1}, \text{ so, } m = \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^{-1} \frac{1-x}{x} \text{ and } \frac{dm}{dx} = -\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^{-1} \frac{1}{x^2} \right]
\end{aligned}$$

$$\begin{aligned}
&= -\frac{d}{a} \int_1^0 a^n \frac{(1-x)^n}{x^n} x^{d+1} \frac{a}{x^2} dx \\
&\quad -\frac{d}{b} \int_1^0 b^n \frac{(1-x)^n}{x^n} x^{d+1} \frac{b}{x^2} dx \\
&\quad -\frac{d}{c} \int_1^0 c^n \frac{(1-x)^n}{x^n} x^{d+1} \frac{c}{x^2} dx \\
&\quad +d \left(\frac{1}{a} + \frac{1}{b}\right) \int_1^0 \left(\frac{1}{a} + \frac{1}{b}\right)^{-n} \frac{(1-x)^n}{x^n} x^{d+1} \left(\frac{1}{a} + \frac{1}{b}\right)^{-1} \frac{1}{x^2} dx \\
&\quad +d \left(\frac{1}{a} + \frac{1}{c}\right) \int_1^0 \left(\frac{1}{a} + \frac{1}{c}\right)^{-n} \frac{(1-x)^n}{x^n} x^{d+1} \left(\frac{1}{a} + \frac{1}{c}\right)^{-1} \frac{1}{x^2} dx \\
&\quad +d \left(\frac{1}{b} + \frac{1}{c}\right) \int_1^0 \left(\frac{1}{b} + \frac{1}{c}\right)^{-n} \frac{(1-x)^n}{x^n} x^{d+1} \left(\frac{1}{b} + \frac{1}{c}\right)^{-1} \frac{1}{x^2} dx \\
&\quad -d \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \int_1^0 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^{-n} \frac{(1-x)^n}{x^n} x^{d+1} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^{-1} \frac{1}{x^2} dx \\
&= \frac{d}{a} a^{n+1} \int_0^1 x^{d-n-1} (1-x)^n dx + \frac{d}{b} b^{n+1} \int_0^1 x^{d-n-1} (1-x)^n dx \\
&\quad + \frac{d}{c} c^{n+1} \int_0^1 x^{d-n-1} (1-x)^n dx - d \left(\frac{1}{a} + \frac{1}{b}\right) \left(\frac{1}{a} + \frac{1}{b}\right)^{-n-1} \int_0^1 x^{d-n-1} (1-x)^n dx \\
&\quad -d \left(\frac{1}{a} + \frac{1}{c}\right) \left(\frac{1}{a} + \frac{1}{c}\right)^{-n-1} \int_0^1 x^{d-n-1} (1-x)^n dx \\
&\quad -d \left(\frac{1}{b} + \frac{1}{c}\right) \left(\frac{1}{b} + \frac{1}{c}\right)^{-n-1} \int_0^1 x^{d-n-1} (1-x)^n dx \\
&\quad +d \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^{-n-1} \int_0^1 x^{d-n-1} (1-x)^n dx \\
&= da^n B(d-n, n+1) + db^n B(d-n, n+1) \\
&\quad +dc^n B(d-n, n+1) - d \left(\frac{1}{a} + \frac{1}{b}\right)^{-n} B(d-n, n+1) \\
&\quad -d \left(\frac{1}{a} + \frac{1}{c}\right)^{-n} B(d-n, n+1) \\
&\quad -d \left(\frac{1}{b} + \frac{1}{c}\right)^{-n} B(d-n, n+1) \\
&\quad +d \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^{-n} B(d-n, n+1).
\end{aligned}$$

d) The cdf of $L = \min(X, Y, Z)$ is

$$\begin{aligned} F_L(l) &= 1 - \Pr(L > l) \\ &= 1 - \Pr(X > l, Y > l, Z > l) \\ &= 1 - \left[1 + \frac{l}{a} + \frac{l}{b} + \frac{l}{c}\right]^{-d}. \end{aligned}$$

e) Differentiating $F_L(l)$ with respect to l gives the pdf of L as

$$f_L(l) = d \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \left[1 + \frac{l}{a} + \frac{l}{b} + \frac{l}{c}\right]^{-d-1}.$$

f) The n th moment of L can be calculated as

$$\begin{aligned} E(L^n) &= d \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \int_0^\infty l^n \left[1 + \frac{l}{a} + \frac{l}{b} + \frac{l}{c}\right]^{-d-1} dl \\ &= d \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^{-n} B(d - n, n + 1). \end{aligned}$$

Solutions to Question 7 (i) The cdf of X is

$$\begin{aligned}
 F_Y(y) &= \Pr(Y \leq y) \\
 &= \Pr(\max(X_1, \dots, X_\alpha) \leq y) \\
 &= \Pr(X_1 \leq y, \dots, X_\alpha \leq y) \\
 &= \Pr(X_1 \leq y) \cdots \Pr(X_\alpha \leq y) \\
 &= [1 - \exp(-\lambda y)] \cdots [1 - \exp(-\lambda y)] \\
 &= [1 - \exp(-\lambda y)]^\alpha.
 \end{aligned}$$

(ii) Differentiating

$$F_Y(y) = [1 - \exp(-\lambda y)]^\alpha$$

with respect to y gives the pdf of Y as

$$f_Y(y) = \alpha \lambda \exp(-\lambda y) [1 - \exp(-\lambda y)]^{\alpha-1}.$$

(iii) The n th moment of Y can be calculated as

$$\begin{aligned}
 E(Y^n) &= \alpha \lambda \int_0^\infty x^n \exp(-\lambda x) [1 - \exp(-\lambda x)]^{\alpha-1} dx \\
 &= (-1)^n \alpha \lambda^{-n} \int_0^1 (\log y)^n [1 - y]^{\alpha-1} dy \\
 &= (-1)^n \alpha \lambda^{-n} \frac{\partial^n}{\partial \beta^n} \int_0^1 y^\beta [1 - y]^{\alpha-1} dy \Big|_{\beta=0} \\
 &= (-1)^n \alpha \lambda^{-n} \frac{\partial^n}{\partial \beta^n} B(\beta + 1, \alpha) \Big|_{\beta=0}.
 \end{aligned}$$

(iv) Setting

$$[1 - \exp(-\lambda y)]^\alpha = p$$

gives

$$\text{VaR}_p(Y) = -\frac{1}{\lambda} \log(1 - p^{1/\alpha}).$$

(v) The expected shortfall is

$$\begin{aligned}
 \text{ES}_p(Y) &= -\frac{1}{\lambda p} \int_0^p \log(1 - v^{1/\alpha}) dv \\
 &= \frac{1}{\lambda p} \int_0^p \sum_{i=0}^{\infty} \frac{v^{i/\alpha}}{i} dv \\
 &= \frac{1}{\lambda p} \sum_{i=0}^{\infty} \int_0^p \frac{v^{i/\alpha}}{i} dv \\
 &= \frac{1}{\lambda p} \sum_{i=0}^{\infty} \frac{p^{i/\alpha+1}}{i(i/\alpha + 1)}.
 \end{aligned}$$

(vi) The likelihood function is

$$L(\alpha, \lambda) = \alpha^n \lambda^n \exp\left(-\lambda \sum_{i=1}^n y_i\right) \left\{ \prod_{i=1}^n [1 - \exp(-\lambda y_i)] \right\}^{\alpha-1}.$$

The log-likelihood function is

$$\log L = n \log(\alpha \lambda) - \lambda \sum_{i=1}^n y_i + (\alpha - 1) \sum_{i=1}^n \log [1 - \exp(-\lambda y_i)].$$

The partial derivatives with respect to α and λ are

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \log [1 - \exp(-\lambda y_i)]$$

and

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^n y_i + (\alpha - 1) \sum_{i=1}^n \frac{y_i \exp(-\lambda y_i)}{1 - \exp(-\lambda y_i)}.$$

Setting these to zero, we obtain the mle of α as

$$\hat{\alpha} = -\frac{n}{\sum_{i=1}^n \log [1 - \exp(-\lambda y_i)]}.$$

The mle of λ is the root of the equation

$$\frac{n}{\lambda} - \sum_{i=1}^n y_i - \left(\frac{n}{\sum_{i=1}^n \log [1 - \exp(-\lambda y_i)]} + 1 \right) \sum_{i=1}^n \frac{y_i \exp(-\lambda y_i)}{1 - \exp(-\lambda y_i)} = 0.$$

Solutions to Question 8 a) If $X_i \sim \text{Exp}(\lambda_i)$, $i = 1, 2$ are independent random variables then

$$\begin{aligned}
 \Pr(X_1 < X_2) &= \int_0^{\infty} [1 - \exp(-\lambda_1 x_2)] \lambda_2 \exp(-\lambda_2 x_2) dx_2 \\
 &= 1 - \int_0^{\infty} \lambda_2 \exp(-\lambda_1 x_2 - \lambda_2 x_2) dx_2 \\
 &= 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} \\
 &= \frac{\lambda_1}{\lambda_1 + \lambda_2}.
 \end{aligned}$$

b) If $X_i \sim \text{Exp}(\lambda_i)$, $i = 1, 2, 3$ are independent random variables then

$$\begin{aligned}
 \Pr(X_1 < X_2 < X_3) &= \int_0^{\infty} \int_{x_1}^{\infty} \int_{x_2}^{\infty} \lambda_1 \lambda_2 \lambda_3 \exp(-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_3) dx_3 dx_2 dx_1 \\
 &= \int_0^{\infty} \int_{x_1}^{\infty} \lambda_1 \lambda_2 \exp(-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_2) dx_2 dx_1 \\
 &= \frac{\lambda_1 \lambda_2}{\lambda_2 + \lambda_3} \int_0^{\infty} \exp(-\lambda_1 x_1 - \lambda_2 x_1 - \lambda_3 x_1) dx_1 \\
 &= \frac{\lambda_1 \lambda_2}{(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3)}.
 \end{aligned}$$

c) If $X_i \sim \text{Exp}(\lambda_i)$, $i = 1, 2, 3, 4$ are independent random variables then

$$\begin{aligned}
 &\Pr(X_1 < X_2 < X_3 < X_4) \\
 &= \int_0^{\infty} \int_{x_1}^{\infty} \int_{x_2}^{\infty} \int_{x_3}^{\infty} \lambda_1 \lambda_2 \lambda_3 \lambda_4 \exp(-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_3 - \lambda_4 x_4) dx_4 dx_3 dx_2 dx_1 \\
 &= \int_0^{\infty} \int_{x_1}^{\infty} \int_{x_2}^{\infty} \lambda_1 \lambda_2 \lambda_3 \exp(-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_3 - \lambda_4 x_3) dx_3 dx_2 dx_1 \\
 &= \frac{\lambda_1 \lambda_2 \lambda_3}{(\lambda_3 + \lambda_4)} \int_0^{\infty} \int_{x_1}^{\infty} \exp(-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_2 - \lambda_4 x_2) dx_2 dx_1 \\
 &= \frac{\lambda_1 \lambda_2 \lambda_3}{(\lambda_3 + \lambda_4)(\lambda_2 + \lambda_3 + \lambda_4)} \int_0^{\infty} \exp(-\lambda_1 x_1 - \lambda_2 x_1 - \lambda_3 x_1 - \lambda_4 x_1) dx_1 \\
 &= \frac{\lambda_1 \lambda_2 \lambda_3}{(\lambda_3 + \lambda_4)(\lambda_2 + \lambda_3 + \lambda_4)(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)}.
 \end{aligned}$$

If $X_i \sim \text{Exp}(\lambda_i)$, $i = 1, 2, \dots, k$ are independent random variables then the general formula will be

$$\Pr(X_1 < X_2 < \dots < X_k) = \frac{\lambda_1 \lambda_2 \dots \lambda_k}{(\lambda_k + \lambda_{k-1})(\lambda_k + \lambda_{k-1} + \lambda_{k-2}) \dots (\lambda_k + \lambda_{k-1} + \dots + \lambda_1)}.$$

This can be proved by induction.

If $\lambda_i = \lambda$ then

$$\Pr(X_1 < X_2) = \frac{1}{2},$$

$$\Pr(X_1 < X_2 < X_3) = \frac{2}{2 \cdot 3},$$

$$\Pr(X_1 < X_2 < X_3 < X_4) = \frac{1}{2 \cdot 3 \cdot 4}$$

and

$$\Pr(X_1 < X_2 < \cdots < X_k) = \frac{1}{2 \cdot 3 \cdots k} = \frac{1}{k!}.$$