Three hours

This exam will be worth 80% of the final mark on this course unit.

THE UNIVERSITY OF MANCHESTER

EXTREME VALUES AND FINANCIAL RISK

26 January 2022

Exam Released:	**9:00 (GMT)**
End of Submission Window:	**16:00 (GMT)**

Answer any **TWO** out of the three questions in Section A.
Answer any **FOUR** of the five questions in Section B.

If more than TWO questions are attempted from Section A then credit will be given to the best TWO answers. If more than FOUR questions are attempted from Section B then credit will be given to the best FOUR answers.

University approved calculators may be used.

SECTION A

Answer any **TWO** questions

A1. Suppose (X, Y) has the joint cumulative distribution function specified by

$$F_{X,Y}(x, y) = [H_{U,V}(x, y)]^{\alpha}$$

for $x > 0$, $y > 0$ and $\alpha > 0$, where H is a valid joint cumulative distribution function of (U, V).

(a) Find the marginal cumulative distribution functions of X and Y, that is $F_X(\cdot)$ and $F_Y(\cdot)$. [**2** marks]

(b) Show that $w(F_X) = w(H_U)$, where $H_U(u) = H_{U,V}(u, \infty)$. [**1** marks]

(c) Show that $w(F_Y) = w(H_V)$, where $H_V(v) = H_{U,V}(\infty, v)$. [**1** marks]

(d) If $H_U(u) = H_{U,V}(u, \infty)$ belongs to the Gumbel max domain of attraction show that $F_X(x)$ also belongs to the Gumbel max domain of attraction. [**3** marks]

(e) If $H_V(v) = H_{U,V}(\infty, v)$ belongs to the Gumbel max domain of attraction show that $F_Y(y)$ also belongs to the Gumbel max domain of attraction. [**3** marks]

(f) If a_n and b_n satisfy

$$[H_U(a_n u + b_n)]^n \to \exp(u)$$

as $n \to \infty$ show that

$$[F_X(a_{\alpha n} x + b_{\alpha n})]^n \to \exp(x)$$

as $n \to \infty$. [**2** marks]

(g) If c_n and d_n satisfy

$$[H_V(c_n v + d_n)]^n \to \exp(v)$$

as $n \to \infty$ show that

$$[F_Y(c_{\alpha n} y + d_{\alpha n})]^n \to \exp(y)$$

as $n \to \infty$. [**2** marks]

(h) If a_n, b_n, c_n and d_n satisfy

$$[H_{U,V}(a_n u + b_n, c_n v + d_n)]^n \to G(x, y)$$

as $n \to \infty$ find the limiting cumulative distribution function $[F_{X,Y}(a_{\alpha n} x + b_{\alpha n}, c_{\alpha n} y + d_{\alpha n})]^n$ as $n \to \infty$. [**5** marks]

(i) Show that the extremes of (X, Y) are completely independent if and only if the extremes of (U, V) completely independent. [**1** marks]

(Total marks: 20)

A2. State the conditions in full for $C(u_1, u_2)$, $0 \leq u_1, u_2 \leq 1$, to be a copula. [**4** marks]

Show that each of the following is a copula function.

(a) the copula defined by

$$C(u_1, u_2) = \min\left[C_1(u_1, u_2), C_2(u_1, u_2)\right],$$

where C_1 and C_2 are valid copulas. [**4** marks]

(b) the copula defined by

$$C(u_1, u_2) = \max\left[C_1(u_1, u_2), C_2(u_1, u_2)\right],$$

where C_1 and C_2 are valid copulas. [**4** marks]

(c) the copula defined by

$$C(u_1, u_2) = \sum_{i=1}^{\infty} \alpha_i C_i(u_1, u_2),$$

where C_i are valid copulas and α_i are non-negative real numbers summing to 1. [**4** marks]

(d) the copula defined by

$$C(u_1, u_2) = \prod_{i=1}^{\infty} \left[C_i(u_1, u_2)\right]^{\alpha_i},$$

where C_i are valid copulas and α_i are non-negative real numbers summing to 1. [**4** marks]

(Total marks: 20)

A3. Consider a bivariate distribution specified by the joint survival function

$$\overline{G}(x, y) = \exp\left\{-(x + y)\sum_{i=1}^{\infty}\alpha_i A_i\left(\frac{y}{x + y}\right)\right\}$$

for $x > 0$ and $y > 0$, where A_i, $i = 1, 2, \ldots$ are convex functions on $[0, 1]$ satisfying $A_i(0) = 1$, $A_i(1) = 1$ and $\max(w, 1 - w) \leq A_i(w) \leq 1$ for all w.

(a) Show that the distribution is a bivariate extreme value distribution. [**7** marks]

(b) Derive the joint cumulative distribution function. [**1** marks]

(c) Derive the conditional cumulative distribution function of Y given $X = x$. You may express this in terms of $A_i^{'}(w)$, the first derivative of $A_i(w)$. [**4** marks]

(d) Derive the conditional cumulative distribution function of X given $Y = y$. You may express this in terms of $A_i^{'}(w)$, the first derivative of $A_i(w)$. [**4** marks]

(e) Derive the joint probability density function. You may express this in terms of $A_i^{'}(w)$ and $A_i^{''}(w)$, the first and second derivatives of $A_i(w)$. [**4** marks]

(Total marks: 20)

P.T.O.

SECTION B

Answer any **FOUR** questions

B1. (a) Suppose $X_1, X_2, \ldots, X_n$ is a random sample with cumulative distribution function $F(\cdot)$. State the Extremal Types Theorem for $M_n = \max(X_1, X_2, \ldots, X_n)$. You must clearly specify the cumulative distribution function of each of the three extreme value distributions. [**4** marks]

(b) State in full the necessary and sufficient conditions for $F(\cdot)$ to belong to the domain of attraction of each of the three extreme value distributions. [**4** marks]

(c) Consider a class of distributions defined by the cumulative distribution function

$$F(x) = 1 - \left[1 - [G(x)]^a \exp\left\{a\left[1 - G(x)\right]\right\}\right]^b$$

where $a > 0$, $b > 0$ and $G(\cdot)$ is a valid cumulative distribution function. Show that F belongs to the same max domain of attraction as G. You may assume that F and G have the same upper end points. [**12** marks]

(Total marks: 20)

B2. Determine the domain of attraction (if there is one) for each of the following distributions:

(a) The distribution given by the cumulative distribution function

$$F(x) = \exp\left[-\exp(-x)\right]$$

for $-\infty < x < \infty$. [**4** marks]

(b) The distribution given by the cumulative distribution function

$$F(x) = \exp\left(-x^{-\alpha}\right)$$

for $x > 0$ and $\alpha > 0$. [**4** marks]

(c) The distribution given by the probability density function

$$f(x) = \frac{Cx^{a-1}}{(1+x)^{a+b}}$$

for $x > 0$, $a > 0$ and $b > 0$, where C is a constant. [**4** marks]

(d) The distribution given by the probability density function

$$f(x) = C\left(1 + ax^2\right)^{-1}\left(1 + bx^2\right)^{-1}$$

for $-\infty < x < \infty$, $a > 0$ and $b > 0$, where C is a constant. [**4** marks]

(e) The distribution given by the probability mass function

$$p(k) = \frac{1}{N}$$

for $k = 1, 2, \ldots, N$. [**4** marks]

(Total marks: 20)

B3. (a) If X is an absolutely continuous random variable with cumulative distribution function $F(\cdot)$, then define $\text{VaR}_p(X)$, the Value at Risk of X, and $\text{ES}_p(X)$, the Expected Shortfall of X, explicitly. [**2** marks]

(b) Suppose X and Y are losses of two investments with joint probability density function

$$f_{X,Y}(x,y) = (a+1)(a+2)(1-x-y)^a$$

for $x > 0$, $y > 0$, $x + y < 1$ and $a > 0$.

(i) Show that the probability density function of $S = X + Y$ is

$$f_S(s) = (a+1)(a+2)s(1-s)^a$$

for $0 < s < 1$. [**2** marks]

(ii) Derive the corresponding cumulative distribution function of S. [**4** marks]

(iii) Show that $\text{VaR}_u(S)$ is the root of

$$(a+2)(1-s)^{a+1} - (a+1)(1-s)^{a+2} = 1-u.$$

[**2** marks]

(iv) Derive the corresponding $\text{ES}_u(S)$. [**5** marks]

(v) If $s_1, \ldots, s_n$ is a random sample on S derive an explicit expressions for the maximum likelihood estimator of a. [**5** marks]

(Total marks: 20)

B4. Suppose X and Y are losses of two investments with joint survival function

$$\overline{F}_{X,Y}(x,y) = \left[\frac{K}{\min(x,y)}\right]^a$$

for $x > K$, $y > K$, $K > 0$ and $a > 0$.

(i) Show that the cumulative distribution function of $U = \min(X,Y)$ is

$$F_U(u) = 1 - \left(\frac{K}{u}\right)^a$$

for $u > K$. [**2** marks]

(ii) Derive the probability density function of U. [**1** marks]

(iii) Derive the mth moment of U. [**4** marks]

(iv) Derive the mean of U. [**1** marks]

(v) Derive the variance of U. [**1** marks]

(vi) Derive the value at risk of U. [**2** marks]

(vii) Derive the expected shortfall of U. [**4** marks]

(viii) If $u_1, \ldots, u_n$ is a random sample on U derive explicit expression for the maximum likelihood estimators of K and a. [**4** marks]

(ix) Deduce the maximum likelihood estimators of the value at risk of U and the expected shortfall of U. [**1** marks]

(Total marks: 20)

B5. Suppose X and Y are losses of two investments with joint cumulative distribution function

$$F_{X,Y}(x,y) = \left[\frac{\max(x,y)}{K}\right]^a$$

for $0 < x \leq K$, $0 < y \leq K$, $K > 0$ and $a > 0$.

(i) Show that the cumulative distribution function of $V = \max(X,Y)$ is

$$F_V(v) = \left(\frac{v}{K}\right)^a$$

for $0 < v < K$. [**2** marks]

(ii) Derive the probability density function of V. [**1** marks]

(iii) Derive the mth moment of V. [**4** marks]

(iv) Derive the mean of V. [**1** marks]

(v) Derive the variance of V. [**1** marks]

(vi) Derive the value at risk of V. [**2** marks]

(vii) Derive the expected shortfall of V. [**4** marks]

(viii) If $v_1, \ldots, v_n$ is a random sample on V derive explicit expression for the maximum likelihood estimators of K and a. [**4** marks]

(ix) Deduce the maximum likelihood estimators of the value at risk of V and the expected shortfall of V. [**1** marks]

(Total marks: 20)

END OF EXAMINATION PAPER