Three hours

This exam will be worth 80% of the final mark on this course unit.

THE UNIVERSITY OF MANCHESTER

EXTREME VALUES AND FINANCIAL RISK

26 January 2021

Exam Released: 9:00 (GMT)
End of Submission Window: 16:00 (GMT)

Answer any **TWO** out of the three questions in Section A.
Answer any **FOUR** of the five questions in Section B.

If more than TWO questions are attempted from Section A then credit will be given to the best TWO answers. If more than FOUR questions are attempted from Section B then credit will be given to the best FOUR answers.

University approved calculators may be used.

SECTION A

Answer any **TWO** questions

A1. Suppose (X, Y) has the joint probability density function specified by

$$f_{X,Y}(x, y) = x + y$$

for $0 < x < 1$ and $0 < y < 1$.

(a) Find the joint cumulative distribution function of X and Y, that is $F_{X,Y}(\cdot, \cdot)$; [**3** marks]

(b) Find the marginal cumulative distribution functions of X and Y, that is $F_X(\cdot)$ and $F_Y(\cdot)$; [**2** marks]

(c) Show that F_X belongs to the Weibull max domain of attraction; [**2** marks]

(d) Show that F_Y also belongs to the Weibull max domain of attraction; [**2** marks]

(e) Find a_n and b_n such that

$$F_X^n(a_n x + b_n) \to \exp(x)$$

as $n \to \infty$; [**2** marks]

(f) Find c_n and d_n such that

$$F_Y^n(c_n x + d_n) \to \exp(x)$$

as $n \to \infty$; [**2** marks]

(g) Find the limiting cumulative distribution function of $F_{X,Y}^n(a_n x + b_n, c_n y + d_n)$ as $n \to \infty$; [**5** marks]

(h) Are the extremes of (X, Y) completely independent? Justify your answer. [**2** marks]

(Total marks: 20)

A2. State the conditions in full for $C(u_1, u_2)$, $0 \leq u_1, u_2 \leq 1$, to be a copula. [**4** marks]

Show that each of the following is a copula function.

(a) the copula defined by $C(u_1, u_2) = \min(u_1, u_2)$. [**4** marks]

(b) the copula defined by

$$C(u_1, u_2) = u_1 u_2 \exp\left[-\theta \log u_1 \log u_2\right]$$

for $0 < \theta \leq 1$. [**4** marks]

(c) the Farlie-Gumbel-Morgenstern copula defined by

$$C(u_1, u_2) = u_1 u_2 \left[1 + \phi (1 - u_1)(1 - u_2)\right]$$

for $-1 \leq \phi \leq 1$. [**4** marks]

(d) the Burr copula defined by

$$C(u_1, u_2) = u_1 + u_2 - 1 + \left[(1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1\right]^{-\alpha}$$

for $\alpha > 0$. [**4** marks]

(Total marks: 20)

A3. Consider a bivariate distribution specified by the joint survival function

$$\overline{F}(x, y) = \Pr(X > x, Y > y) = \exp\left[-\frac{\theta y^2}{x + y} + \theta y - x - y\right]$$

for $x > 0$ and $y > 0$.

(a) Show that the distribution is a bivariate extreme value distribution. [**6** marks]

(b) Derive the joint cumulative distribution function. [**2** marks]

(c) Derive the conditional cumulative distribution function of Y given $X = x$. [**4** marks]

(d) Derive the conditional cumulative distribution function of X given $Y = y$. [**4** marks]

(e) Derive the joint probability density function. [**4** marks]

(Total marks: 20)

SECTION B

Answer any **FOUR** questions

B1. (a) Suppose $X_1, X_2, \ldots, X_n$ is a random sample with cumulative distribution function $F(\cdot)$. State the Extremal Types Theorem for $M_n = \max(X_1, X_2, \ldots, X_n)$. You must clearly specify the cumulative distribution function of each of the three extreme value distributions. [**4** marks]

(b) State in full the necessary and sufficient conditions for $F(\cdot)$ to belong to the domain of attraction of each of the three extreme value distributions. [**4** marks]

(c) Consider a class of distributions defined by the cumulative distribution function

$$F(x) = \frac{aG(x)\,[1 + b - bG(x)]}{1 - (1 - a)G(x)\,[1 + b - bG(x)]}$$

where $a > 0$, $b > 0$ and $G(\cdot)$ is a valid cumulative distribution function. Show that F belongs to the same max domain of attraction as G. You may assume that F and G have the same upper end points. [**12** marks]

(Total marks: 20)

B2. Determine the domain of attraction (if there is one) for each of the following distributions:

(a) The distribution given by the probability mass function

$$p(k) = a(1-a)^{k-1}$$

for $0 < a < 1$ and $k = 1, 2, \ldots$; [**4** marks]

(b) The distribution given by the cumulative distribution function

$$F(x) = \frac{2}{\pi}\arcsin\left(\sqrt{x}\right)$$

for $-\infty < x < \infty$; [**4** marks]

(c) The distribution given by the cumulative distribution function

$$F(x) = \frac{x^b}{a^b + x^b}$$

for $x > 0$, $a > 0$ and $b > 0$; [**4** marks]

(d) The distribution given by the probability density function

$$f(x) = C\,(x-a)^{-\frac{3}{2}} \exp\left[-\frac{1}{2(x-a)}\right]$$

for $x > a > 0$, where C is a constant; [**4** marks]

(e) The distribution given by the probability density function

$$f(x) = C\left(1 + \frac{x^2}{a}\right)^{-\frac{a+1}{2}}$$

for $-\infty < x < \infty$ and $a > 0$, where C is a constant. [**4** marks]

(Total marks: 20)

B3. (a) If X is an absolutely continuous random variable with cumulative distribution function $F(\cdot)$, then define $\mathrm{VaR}_p(X)$, the Value at Risk of X, and $\mathrm{ES}_p(X)$, the Expected Shortfall of X, explicitly. [**2** marks]

(b) Suppose X is a random variable with probability density function given by

$$f(x) = \frac{3x^2}{2a^3}$$

for $-a < x < a$.

(i) Show that the corresponding cumulative distribution function is

$$F(x) = \frac{x^3 + a^3}{2a^3}$$

for $-a < x < a$; [**2** marks]

(ii) Derive the corresponding $\mathrm{VaR}_p(X)$; [**1** marks]

(iii) Derive the corresponding $\mathrm{ES}_p(X)$. [**2** marks]

(c) Suppose $X_1, X_2, \ldots, X_n$ is a random sample on X in (b).

(i) Write down the likelihood function of a; [**3** marks]

(ii) Show that the maximum likelihood estimator of a is

$$\widehat{a} = \max\left[\max\left(X_1, X_2, \ldots, X_n\right), -\min\left(X_1, X_2, \ldots, X_n\right)\right];$$

[**1** marks]

(iii) Deduce the maximum likelihood estimators of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$; [**2** marks]

(iv) Let $Z = \max\left[\max\left(X_1, X_2, \ldots, X_n\right), -\min\left(X_1, X_2, \ldots, X_n\right)\right]$. Show that the cumulative distribution function of Z is

$$F_Z(z) = \left[\frac{z^3}{a^3}\right]^n$$

for $z > 0$; [**3** marks]

(v) Hence, show that the maximum likelihood estimator $\widehat{a}$ is biased but consistent for a; [**2** marks]

(vi) Hence, show that the maximum likelihood estimator of $\mathrm{VaR}_p(X)$ is also biased but consistent for $\mathrm{VaR}_p(X)$; [**1** marks]

(vii) Hence, show that the maximum likelihood estimator of $\mathrm{ES}_p(X)$ is also biased but consistent for $\mathrm{ES}_p(X)$. [**1** marks]

(Total marks: 20)

B4. (a) Suppose $X_1, X_2, \ldots, X_k$ are losses on k investments. Suppose also $X_1, X_2, \ldots, X_k$ are independent $N\left(\mu_i, \sigma_i^2\right)$, $i = 1, 2, \ldots, k$ random variables. Let $T = X_1 + X_2 + \cdots + X_k$ denote the total loss.

(i) Determine the distribution of T; [**2** marks]

(ii) Derive the corresponding $\mathrm{VaR}_p(T)$; [**2** marks]

(iii) Derive the corresponding $\mathrm{ES}_p(T)$. [**2** marks]

(b) Suppose $X_{i,1}, X_{i,2}, \ldots, X_{i,n}$ is a random sample on X_i in (a). Suppose also $X_{i,1}, X_{i,2}, \ldots, X_{i,n}$ and $X_{j,1}, X_{j,2}, \ldots, X_{j,n}$ are independent for $i \neq j$.

(i) Write down the joint likelihood function of $\mu_1, \mu_2, \ldots, \mu_k$ and $\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2$; [**2** marks]

(ii) Show that the maximum likelihood estimators are

$$\widehat{\mu_i} = \frac{1}{n} \sum_{j=1}^{n} X_{i,j}$$

and

$$\widehat{\sigma_i^2} = \frac{1}{n} \sum_{j=1}^{n} \left(X_{i,j} - \widehat{\mu_i}\right)^2;$$

[**4** marks]

(iii) Show that $\widehat{\mu_i}$ is unbiased and consistent for μ_i; [**3** marks]

(iv) Show that $\widehat{\sigma_i^2}$ is biased and consistent for σ_i^2; [**3** marks]

(v) Deduce the maximum likelihood estimators of $\mathrm{VaR}_p(T)$ and $\mathrm{ES}_p(T)$. [**2** marks]

(Total marks: 20)

B5. Suppose that a portfolio is made up of k investments where k is known. Suppose also that the losses on the investments, say X_i, $i = 1, 2, \ldots, k$, are dependent random variables with joint survival function

$$\overline{F}\left(x_1, x_2, \ldots, x_k\right) = \exp\left[-\sum_{i=1}^{k} x_i - \lambda \max\left(x_1, x_2, \ldots, x_k\right)\right]$$

for $\lambda > 0$ and $x_i > 0$, $i = 1, 2, \ldots, k$, where λ is an unknown parameter.

(a) Show that the cumulative distribution function of $V = \min\left(X_1, X_2, \ldots, X_k\right)$, the minimum portfolio loss, is

$$F_V\left(v\right) = 1 - \exp\left(-kv - \lambda v\right);$$

[**6** marks]

(b) Derive the corresponding probability density function of V; [**2** marks]

(c) Derive the corresponding $\mathrm{VaR}_p(V)$; [**2** marks]

(d) Derive the corresponding $\mathrm{ES}_p(V)$; [**2** marks]

(e) Let $v_1, v_2, \ldots, v_n$ be a random sample on V. Derive the maximum likelihood estimator of λ. [**8** marks]

(Total marks: 20)

END OF EXAMINATION PAPER