MATH68181

<u>**Three hours**</u>

To be supplied by the Examinations Office: Mathematical Formula Tables and Statistical Tables

THE UNIVERSITY OF MANCHESTER

EXTREME VALUES AND FINANCIAL RISK

13 January 2020

9:45am-12:45pm

Answer any **TWO** out of the three questions in Section A.
Answer any **FOUR** of the five questions in Section B.

If more than TWO questions are attempted from Section A then credit will be given to the best TWO answers. If more than FOUR questions are attempted from Section B then credit will be given to the best FOUR answers.

University approved calculators may be used.

SECTION A

Answer any **TWO** questions

A1. Suppose (X, Y) has the joint survival function specified by

$$\overline{F}_{X,Y}(x, y) = \exp\left[-ax - by - c\max(x, y)\right]$$

for $x > 0$, $y > 0$, $a > 0$, $b > 0$ and $c > 0$.

(a) Find the joint cumulative distribution function of X and Y, that is $F_{X,Y}(\cdot, \cdot)$; [**3** marks]

(b) Find the marginal cumulative distribution functions of X and Y, that is $F_X(\cdot)$ and $F_Y(\cdot)$; [**2** marks]

(c) Show that F_X belongs to the Gumbel max domain of attraction; [**2** marks]

(d) Show that F_Y also belongs to the Gumbel max domain of attraction; [**2** marks]

(e) Find a_n and b_n such that

$$F_X^n\left(a_n x + b_n\right) \to \exp(-x)$$

as $n \to \infty$; [**2** marks]

(f) Find c_n and d_n such that

$$F_Y^n\left(c_n x + d_n\right) \to \exp(-x)$$

as $n \to \infty$; [**2** marks]

(g) Find the limiting cumulative distribution function of $F_{X,Y}^n\left(a_n x + b_n, c_n y + d_n\right)$ as $n \to \infty$; [**5** marks]

(h) Are the extremes of (X, Y) completely independent? Justify your answer. [**2** marks]

(Total marks: 20)

A2. State the conditions in full for $C\left(u_1, u_2\right)$, $0 \leq u_1, u_2 \leq 1$ to be a copula. [**4** marks]

Show that each of the following is a copula function.

(a) $C\left(u_1, u_2\right) = \exp\left\{-\left[\left(-\log u_1\right)^\theta + \left(-\log u_2\right)^\theta\right]^{1/\theta}\right\}$ for $\theta > 0$. [**4** marks]

(b) $C\left(u_1, u_2\right) = \begin{cases} \max\left(u_1 + u_2 - 1, t\right), & \text{if } t \leq u_1 \leq 1, t \leq u_2 \leq 1, \\ \min\left(u_1, u_2\right), & \text{otherwise} \end{cases}$ for $0 < t < 1$. [**4** marks]

(c) $C\left(u_1, u_2\right) = \left\{\left[\left(u_1^{-\theta} - 1\right)^\delta + \left(u_2^{-\theta} - 1\right)^\delta\right]^{1/\delta} + 1\right\}^{-1/\theta}$ for $\theta > 0$ and $\delta \geq 1$. [**4** marks]

(d) $C\left(u_1, u_2\right) = w_1 C_1\left(u_1, u_2\right) + w_2 C_2\left(u_1, u_2\right) + \cdots + w_p C_p\left(u_1, u_2\right)$, where $C_1, C_2, \ldots, C_p$ are valid copulas and $w_1, w_2, \ldots, w_p$ are non-negative constants summing to one. [**4** marks]

(Total marks: 20)

A3. Consider a bivariate distribution specified by the joint survival function

$$\overline{G}(x, y) = \exp\left\{-x - y + (\theta + \phi)y - \frac{\theta y^2}{x + y} - \frac{\phi y^3}{(x + y)^2}\right\}$$

for $x > 0$, $y > 0$, $\theta \geq 0$, $\phi \geq 0$, $\theta + 3\phi \geq 0$, $\theta + \phi \leq 1$ and $\theta + 2\phi \leq 1$.

(a) Show that the distribution is a bivariate extreme value distribution; [**6** marks]

(b) Derive the joint cumulative distribution function; [**2** marks]

(c) Derive the conditional cumulative distribution function of Y given $X = x$; [**4** marks]

(d) Derive the conditional cumulative distribution function of X given $Y = y$; [**4** marks]

(e) Derive the joint probability density function. [**4** marks]

(Total marks: 20)

SECTION B

Answer any **FOUR** questions

B1. (a) Suppose $X_1, X_2, \ldots, X_n$ is a random sample with cumulative distribution function $F(\cdot)$. State the Extremal Types Theorem for $M_n = \max(X_1, X_2, \ldots, X_n)$. You must clearly specify the cumulative distribution function of each of the three extreme value distributions. [**4** marks]

(b) State in full the necessary and sufficient conditions for $F(\cdot)$ to belong to the domain of attraction of each of the three extreme value distributions. [**4** marks]

(c) Consider a class of distributions defined by the cumulative distribution function

$$F(x) = 1 - \left\{ 1 - \left[(1+\lambda) G(x) - \lambda \left(G(x) \right)^2 \right]^a \right\}^b$$

where $a > 0$, $b > 0$, $-\infty < \lambda < \infty$ and $G(\cdot)$ is a valid cumulative distribution function. Show that F belongs to the same max domain of attraction as G. You may assume that F and G have the same upper end points. [**12** marks]

(Total marks: 20)

B2. Determine the domain of attraction (if there is one) for each of the following distributions:

(a) The distribution given by the probability mass function

$$p(k) = a(1-a)^{k-1}$$

for $0 < a < 1$ and $k = 1, 2, \ldots$; [**4** marks]

(b) The distribution given by the cumulative distribution function

$$F(x) = \frac{1-\exp(-x)}{1+\exp(-x)}$$

for $x > 0$; [**4** marks]

(c) The distribution given by the probability density function

$$f(x) = Cx^{2a-1}\exp\left(-x^2\right)$$

for $x > 0$ and $a > 0.5$, where C is a constant; [**4** marks]

(d) The distribution given by the cumulative distribution function

$$F(x) = \left[1-\exp\left(-x^a\right)\right]^b$$

for $x > 0$, $a > 0$ and $b > 0$; [**4** marks]

(e) The distribution given by the cumulative distribution function

$$F(x) = 1 - \frac{\log\left[1-(1-p)\exp(-x)\right]}{\log p}$$

for $x > 0$ and $0 < p < 1$. [**4** marks]

(Total marks: 20)

B3. (a) If X is an absolutely continuous random variable with cdf $F(\cdot)$, then define $\mathrm{VaR}_p(X)$, the Value at Risk of X, and $\mathrm{ES}_p(X)$, the Expected Shortfall of X, explicitly. [**2** marks]

(b) Suppose a portfolio is made up of two investments. Suppose also that the losses on the investments say X and Y are dependent random variables with joint survival function

$$\overline{F}(x, y) = \exp(-x - y - \theta xy)$$

for $x > 0$ and $y > 0$, where $0 < \theta < 1$ is an unknown parameter.

(i) Show that the cumulative distribution function of the minimum portfolio loss $V = \min(X, Y)$ is

$$F_V(v) = 1 - \exp\left(-2v - \theta v^2\right)$$

for $v > 0$. [**5** marks]

(ii) Derive the corresponding probability density function of V; [**3** marks]

(iii) Derive the corresponding $\mathrm{VaR}_p(V)$; [**5** marks]

(iv) If $v_1, v_2, \ldots, v_n$ is a random sample on V derive the equation satisfied by the maximum likelihood estimator of θ and show why its solution must be a maximum likelihood estimator. [**5** marks]

(Total marks: 20)

B4. Suppose a portfolio is made up of k investments where k is known. Suppose also that the losses on the investments say X_i, $i = 1, 2, \ldots, k$ are dependent random variables with joint cumulative distribution function

$$F\left(x_1, x_2, \ldots, x_k\right) = \left[\frac{1}{1 + \displaystyle\sum_{i=1}^{k} \exp\left(-x_i\right)}\right]^c$$

for $-\infty < x_i < \infty$, $i = 1, 2, \ldots, k$, where c is an unknown parameter.

(a) Show that the cumulative distribution function the maximum portfolio loss $U = \max\left(X_1, X_2, \ldots, X_k\right)$ is

$$F_U\left(u\right) = \left[\frac{1}{1 + k\exp\left(-u\right)}\right]^c;$$

[**8** marks]

(b) Derive the corresponding probability density function of U; [**2** marks]

(c) Derive the corresponding $\mathrm{VaR}_p(U)$; [**2** marks]

(d) If $u_1, u_2, \ldots, u_n$ is a random sample on U derive the maximum likelihood estimator of c. [**8** marks]

(Total marks: 20)

B5. Suppose a portfolio is made up of two investments. Suppose also that the losses on the investments say X and Y are dependent random variables with joint probability density function

$$f(x, y) = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} x^{a-1} y^{b-1} (1-x-y)^{c-1}$$

for $x > 0$, $y > 0$ and $x + y < 1$, where $a > 0$, $b > 0$ and $c > 0$ are unknown parameters.

(a) Show that the probability density function of the total portfolio loss $S = X + Y$ is

$$f_S(s) = \frac{\Gamma(a+b+c)}{\Gamma(a+b)\Gamma(c)} s^{a+b-1} (1-s)^{c-1}$$

for $0 < s < 1$. [**8** marks]

(b) Derive the corresponding cumulative distribution function of S; [**2** marks]

(c) Derive the corresponding $\mathrm{VaR}_p(S)$; [**2** marks]

(d) If $s_1, s_2, \ldots, s_n$ is a random sample on S derive the equations satisfied by the maximum likelihood estimators of a, b and c. [**8** marks]

(Total marks: 20)

END OF EXAMINATION PAPER