MATH68181 (REPRINT)

Three hours

To be supplied by the Examinations Office: Mathematical Formula Tables and Statistical Tables

THE UNIVERSITY OF MANCHESTER

EXTREME VALUES AND FINANCIAL RISK

23 January 2019

14:00pm-17:00pm

Answer any **TWO** out of the three questions in Section A.
Answer any **FOUR** of the five questions in Section B.

If more than TWO questions are attempted from Section A then credit will be given to the best TWO answers. If more than FOUR questions are attempted from Section B then credit will be given to the best FOUR answers.

Electronic calculators are permitted provided that cannot store text.

SECTION A

Answer any **TWO** questions

A1. Suppose that (X, Y) has the joint survival function specified by

$$\overline{F}_{X,Y}(x, y) = (1 + ax + by + cxy)^{-q}$$

for $x > 0$, $y > 0$, $q > 0$, $a > 0$, $b > 0$ and $0 \leq c \leq ab(1 + q)$.

(a) Find the joint cumulative distribution function of X and Y, that is $F_{X,Y}(\cdot, \cdot)$; [**3** marks]

(b) Find the marginal cumulative distribution functions of X and Y, that is $F_X(\cdot)$ and $F_Y(\cdot)$; [**2** marks]

(c) Show that F_X belongs to the Fréchet max domain of attraction; [**2** marks]

(d) Show that F_Y also belongs to the Fréchet max domain of attraction; [**2** marks]

(e) Find a_n and b_n such that

$$[F_X(a_n x + b_n)]^n \to \exp\left[-x^{-q}\right]$$

as $n \to \infty$; [**2** marks]

(f) Find c_n and d_n such that

$$[F_Y(c_n x + d_n)]^n \to \exp\left[-x^{-q}\right]$$

as $n \to \infty$; [**2** marks]

(g) Find the limiting cumulative distribution function of $[F_{X,Y}(a_n x + b_n, c_n y + d_n)]^n$ as $n \to \infty$; [**5** marks]

(h) Are the extremes of (X, Y) completely independent? Justify your answer. [**2** marks]

(Total marks: 20)

A2. (a) State the conditions in full for $C(u_1, u_2)$, $0 \leq u_1, u_2 \leq 1$ to be a copula. [**4** marks]

(b) Show that each of the following is a copula function.

(i) $C(u_1, u_2) = u_1^{1-a} u_2^{1-b} \min\left(u_1^a, u_2^b\right)$ for $0 < a < 1$ and $0 < b < 1$; [**4** marks]

(ii) $C(u_1, u_2) = \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2}$; [**4** marks]

(iii) $C(u_1, u_2) = \left[C_1(u_1, u_2) C_2(u_1, u_2) \cdots C_p(u_1, u_2)\right]^{1/p}$, where $C_1, C_2, \ldots, C_p$ are valid copulas; [**4** marks]

(iv) $C(u_1, u_2) = w_1 C_1(u_1, u_2) + w_2 C_2(u_1, u_2) + \cdots + w_p C_p(u_1, u_2)$, where $C_1, C_2, \ldots, C_p$ are valid copulas and $w_1, w_2, \ldots, w_p$ are non-negative constants summing to one. [**4** marks]

(Total marks: 20)

A3. Suppose that a random vector (X, Y) has the joint survival function specified by

$$\overline{G}(x, y) = \exp\left\{-x - y + \frac{\alpha xy}{x + y}\left[1 - \frac{\beta xy}{(x + y)^2}\right]\right\}$$

for $x > 0$, $y > 0$, $0 < \alpha \leq 1$ and $0 < \beta \leq 2$.

(a) Show that the distribution is a bivariate extreme value distribution; [**7** marks]

(b) Derive the joint cumulative distribution function of X and Y; [**1** marks]

(c) Derive the conditional cumulative distribution function of Y given $X = x$; [**4** marks]

(d) Derive the conditional cumulative distribution function of X given $Y = y$; [**4** marks]

(e) Derive the joint probability density function of X and Y. [**4** marks]

(Total marks: 20)

SECTION B

Answer any **FOUR** questions

B1. (a) Suppose that $X_1, X_2, \ldots, X_n$ is a random sample with cumulative distribution function $F(\cdot)$. State the Extremal Types Theorem for $M_n = \max(X_1, X_2, \ldots, X_n)$. You must clearly specify the cumulative distribution function of each of the three extreme value distributions. [**4** marks]

(b) State in full the necessary and sufficient conditions for $F(\cdot)$ to belong to the domain of attraction of each of the three extreme value distributions. [**4** marks]

(c) Consider a class of distributions defined by the cumulative distribution function

$$F(x) = \frac{[G(x)]^a}{[G(x)]^a + [1 - G(x)]^a},$$

where $a > 0$ and $G(\cdot)$ is a cumulative distribution function. Show that F belongs to the same max domain of attraction as G. You may assume that F and G have the same upper end points.

[**12** marks]

(Total marks: 20)

B2. Determine the domain of attraction (if there is one) for each of the following distributions:

(a) The distribution given by the probability mass function

$$p(k) = a(1-a)^{k-1}$$

for $0 < a < 1$ and $k = 1, 2, \ldots$; [**4** marks]

(b) The distribution given by the probability mass function

$$p(k) = \begin{cases} p, & \text{if } k = 1, \\ 1-p, & \text{if } k = 0 \end{cases}$$

for $0 < p < 1$; [**4** marks]

(c) The distribution given by the probability density function

$$f(x) = abx^{a-1}\left(1 - x^a\right)^{b-1}$$

for $x > 0$, $a > 0$ and $b > 0$; [**4** marks]

(d) The distribution given by the cumulative distribution function

$$F(x) = \left[1 + x^{-a}\right]^{-b}$$

for $x > 0$, $a > 0$ and $b > 0$; [**4** marks]

(e) The distribution given by the probability density function

$$f(x) = \frac{C}{\sqrt{x(1-x)}}$$

for $0 < x < 1$ and C a constant. [**4** marks]

(Total marks: 20)

B3. (a) If X is an absolutely continuous random variable with cumulative distribution function $F(\cdot)$, then define $\mathrm{VaR}_p(X)$, the Value at Risk of X, and $\mathrm{ES}_p(X)$, the Expected Shortfall of X, explicitly. [**2** marks]

(b) Suppose that a portfolio is made up of k investments where k is known. Suppose also that the losses on the investments, say X_i, $i = 1, 2, \ldots, k$, are dependent random variables with joint probability density function

$$f\left(x_1, x_2, \ldots, x_k\right) = \frac{\Gamma\left(k + \frac{a}{2}\right)}{\Gamma\left(\frac{a}{2}\right)} \left(1 - \sum_{i=1}^{k} x_i\right)^{\frac{a}{2}-1}$$

for $a > 0$, $x_i > 0$, $i = 1, 2, \ldots, k$ and $0 < x_1 + x_2 + \cdots + x_k < 1$.

(i) Show that the probability density function of the total portfolio loss $S = X_1 + \cdots + X_k$ is

$$f_S(s) = \frac{1}{B\left(\frac{a}{2}, k\right)} s^{k-1} (1 - s)^{\frac{a}{2}-1}$$

for $0 < s < 1$. You may use the following identity without proof:

$$\int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{k-2}} dx_{k-1} \cdots dx_2 dx_1 = \frac{s^{k-1}}{(k-1)!};$$

[**6** marks]

(ii) Derive the nth moment of S; [**3** marks]

(iii) Derive the cumulative distribution function of S; [**3** marks]

(iv) Derive the corresponding $\mathrm{VaR}_p(S)$; [**3** marks]

(v) Derive the corresponding $\mathrm{ES}_p(S)$. [**3** marks]

(Total marks: 20)

B4. Suppose that a portfolio is made up of k investments where k is known. Suppose also that the losses on the investments, say X_i, $i = 1, 2, \ldots, k$, are dependent random variables with joint cumulative distribution function

$$F\left(x_1, x_2, \ldots, x_k\right) = \frac{1}{1 + \displaystyle\sum_{i=1}^{k} \exp\left(-x_i\right)}$$

for $-\infty < x_i < \infty$, $i = 1, 2, \ldots, k$.

(a) Show that the cumulative distribution function of $U = \max\left(X_1, X_2, \ldots, X_k\right)$, the maximum portfolio loss, is

$$F_U\left(u\right) = \frac{1}{1 + k \exp\left(-u\right)};$$

[**6** marks]

(b) Derive the corresponding probability density function of U; [**2** marks]

(c) Derive the corresponding moment generating function of U; [**5** marks]

(d) Derive the corresponding $\mathrm{VaR}_p(U)$; [**2** marks]

(e) Derive the corresponding $\mathrm{ES}_p(U)$. [**5** marks]

(Total marks: 20)

B5. Suppose that a portfolio is made up of k investments where k is known. Suppose also that the losses on the investments, say X_i, $i = 1, 2, \ldots, k$, are dependent random variables with joint survival function

$$\overline{F}\left(x_1, x_2, \ldots, x_k\right) = \exp\left[-\sum_{i=1}^{k} x_i - \lambda \max\left(x_1, x_2, \ldots, x_k\right)\right]$$

for $\lambda > 0$ and $x_i > 0$, $i = 1, 2, \ldots, k$, where λ is an unknown parameter.

(a) Show that the cumulative distribution function of $V = \min\left(X_1, X_2, \ldots, X_k\right)$, the minimum portfolio loss, is

$$F_V\left(v\right) = 1 - \exp\left(-kv - \lambda v\right);$$

[**6** marks]

(b) Derive the corresponding probability density function of V; [**2** marks]

(c) Derive the corresponding $\mathrm{VaR}_p(V)$; [**2** marks]

(d) Derive the corresponding $\mathrm{ES}_p(V)$; [**2** marks]

(e) Let $v_1, v_2, \ldots, v_n$ be a random sample on V. Derive the maximum likelihood estimator of λ. [**8** marks]

(Total marks: 20)

END OF EXAMINATION PAPER