Two hours

To be supplied by the Examinations Office: Mathematical Formula Tables and Statistical Tables

THE UNIVERSITY OF MANCHESTER

EXTREME VALUES AND FINANCIAL RISK

23 January 2019

14:00pm-16:00pm

Answer any **FOUR** of the five questions.

If more than FOUR questions are attempted then credit will be given to the best FOUR answers.

Electronic calculators are permitted provided that cannot store text.

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1. (a) Suppose that $X_1, X_2, \ldots, X_n$ is a random sample with cumulative distribution function $F(\cdot)$. State the Extremal Types Theorem for $M_n = \max(X_1, X_2, \ldots, X_n)$. You must clearly specify the cumulative distribution function of each of the three extreme value distributions. [**4** marks]

(b) State in full the necessary and sufficient conditions for $F(\cdot)$ to belong to the domain of attraction of each of the three extreme value distributions. [**4** marks]

(c) Consider a class of distributions defined by the cumulative distribution function

$$F(x) = \frac{[G(x)]^a}{[G(x)]^a + [1 - G(x)]^a},$$

where $a > 0$ and $G(\cdot)$ is a cumulative distribution function. Show that F belongs to the same max domain of attraction as G. You may assume that F and G have the same upper end points.

[**12** marks]

(Total marks: 20)

2. Determine the domain of attraction (if there is one) for each of the following distributions:

(a) The distribution given by the probability mass function

$$p(k) = a(1-a)^{k-1}$$

for $0 < a < 1$ and $k = 1, 2, \ldots$; [**4** marks]

(b) The distribution given by the probability mass function

$$p(k) = \begin{cases} p, & \text{if } k = 1, \\ 1-p, & \text{if } k = 0 \end{cases}$$

for $0 < p < 1$; [**4** marks]

(c) The distribution given by the probability density function

$$f(x) = abx^{a-1}\left(1 - x^a\right)^{b-1}$$

for $x > 0$, $a > 0$ and $b > 0$; [**4** marks]

(d) The distribution given by the cumulative distribution function

$$F(x) = \left[1 + x^{-a}\right]^{-b}$$

for $x > 0$, $a > 0$ and $b > 0$; [**4** marks]

(e) The distribution given by the probability density function

$$f(x) = \frac{C}{\sqrt{x(1-x)}}$$

for $0 < x < 1$ and C a constant. [**4** marks]

(Total marks: 20)

3. (a) If X is an absolutely continuous random variable with cumulative distribution function $F(\cdot)$, then define $\mathrm{VaR}_p(X)$, the Value at Risk of X, and $\mathrm{ES}_p(X)$, the Expected Shortfall of X, explicitly.

[**2** marks]

(b) Suppose that a portfolio is made up of k investments where k is known. Suppose also that the losses on the investments, say X_i, $i = 1, 2, \ldots, k$, are dependent random variables with joint probability density function

$$f\left(x_1, x_2, \ldots, x_k\right) = \frac{\Gamma\left(k + \dfrac{a}{2}\right)}{\Gamma\left(\dfrac{a}{2}\right)} \left(1 - \sum_{i=1}^{k} x_i\right)^{\frac{a}{2}-1}$$

for $a > 0$, $x_i > 0$, $i = 1, 2, \ldots, k$ and $0 < x_1 + x_2 + \cdots + x_k < 1$.

(i) Show that the probability density function of the total portfolio loss $S = X_1 + \cdots + X_k$ is

$$f_S(s) = \frac{1}{B\left(\frac{a}{2}, k\right)} s^{k-1} \left(1 - s\right)^{\frac{a}{2}-1}$$

for $0 < s < 1$. You may use the following identity without proof:

$$\int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{k-2}} dx_{k-1} \cdots dx_2 dx_1 = \frac{s^{k-1}}{(k-1)!};$$

[**6** marks]

(ii) Derive the nth moment of S; [**3** marks]

(iii) Derive the cumulative distribution function of S; [**3** marks]

(iv) Derive the corresponding $\mathrm{VaR}_p(S)$; [**3** marks]

(v) Derive the corresponding $\mathrm{ES}_p(S)$. [**3** marks]

(Total marks: 20)

4. Suppose that a portfolio is made up of k investments where k is known. Suppose also that the losses on the investments, say X_i, $i = 1, 2, \ldots, k$, are dependent random variables with joint cumulative distribution function

$$F(x_1, x_2, \ldots, x_k) = \frac{1}{1 + \displaystyle\sum_{i=1}^{k} \exp(-x_i)}$$

for $-\infty < x_i < \infty$, $i = 1, 2, \ldots, k$.

(a) Show that the cumulative distribution function of $U = \max(X_1, X_2, \ldots, X_k)$, the maximum portfolio loss, is

$$F_U(u) = \frac{1}{1 + k \exp(-u)};$$

[**6** marks]

(b) Derive the corresponding probability density function of U; [**2** marks]

(c) Derive the corresponding moment generating function of U; [**5** marks]

(d) Derive the corresponding $\mathrm{VaR}_p(U)$; [**2** marks]

(e) Derive the corresponding $\mathrm{ES}_p(U)$. [**5** marks]

(Total marks: 20)

5. Suppose that a portfolio is made up of k investments where k is known. Suppose also that the losses on the investments, say X_i, $i = 1, 2, \ldots, k$, are dependent random variables with joint survival function

$$\overline{F}(x_1, x_2, \ldots, x_k) = \exp\left[-\sum_{i=1}^{k} x_i - \lambda \max(x_1, x_2, \ldots, x_k)\right]$$

for $\lambda > 0$ and $x_i > 0$, $i = 1, 2, \ldots, k$, where λ is an unknown parameter.

(a) Show that the cumulative distribution function of $V = \min(X_1, X_2, \ldots, X_k)$, the minimum portfolio loss, is

$$F_V(v) = 1 - \exp(-kv - \lambda v);$$

[**6** marks]

(b) Derive the corresponding probability density function of V; [**2** marks]

(c) Derive the corresponding $\mathrm{VaR}_p(V)$; [**2** marks]

(d) Derive the corresponding $\mathrm{ES}_p(V)$; [**2** marks]

(e) Let $v_1, v_2, \ldots, v_n$ be a random sample on V. Derive the maximum likelihood estimator of λ. [**8** marks]

(Total marks: 20)

END OF EXAMINATION PAPER