MATH48181

Three hours

To be supplied by the Examinations Office: Mathematical Formula Tables and Statistical Tables

THE UNIVERSITY OF MANCHESTER

EXTREME VALUES AND FINANCIAL RISK

Examiner:

24 January 2018

9:45am-12:45pm

Answer ANY TWO questions in Section A.
Answer ANY FOUR questions in Section B.

Electronic calculators are permitted provided that cannot store text.

SECTION A

Answer any **TWO** questions

A1. Suppose (X, Y) has the joint pdf specified by

$$f_{X,Y}(x, y) = \frac{4xy + 2x + 2y + 1}{4}$$

for $0 < x < 1$ and $0 < y < 1$.

(a) Find the joint cdf of X and Y, that is $F_{X,Y}(\cdot, \cdot)$; (3 marks)

(b) Find the marginal cdfs of X and Y, that is $F_X(\cdot)$ and $F_Y(\cdot)$; (2 marks)

(c) Show that F_X belongs to the Weibull max domain of attraction; (2 marks)

(d) Show that F_Y also belongs to the Weibull max domain of attraction; (2 marks)

(e) Find a_n and b_n such that

$$F_X^n\left(a_n x + b_n\right) \to \exp(x)$$

as $n \to \infty$; (2 marks)

(f) Find c_n and d_n such that

$$F_Y^n\left(c_n x + d_n\right) \to \exp(x)$$

as $n \to \infty$; (2 marks)

(g) Find the limiting cdf of $F_{X,Y}^n\left(a_n x + b_n, c_n y + d_n\right)$ as $n \to \infty$; (5 marks)

(h) Are the extremes of (X, Y) completely independent? Justify your answer. (2 marks)

(Total marks: 20)

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A2. State the conditions in full for $C(u_1, u_2)$, $0 \le u_1, u_2 \le 1$ to be a copula. (4 marks)

Show that each of the following is a copula function.

(a) $C(u_1, u_2) = \min\left(u_1^{1-\alpha} u_2, u_1 u_2^{1-\beta}\right)$ for $0 < \alpha < 1$ and $0 < \beta < 1$;
(4 marks)

(b) $C(u_1, u_2) = u_2 - \left[\max\left((1-u_1)^{1/n} + u_2^{1/n} - 1, 0\right)\right]^n$ for $n \ge 1$ an integer; (4 marks)

(c) $C(u_1, u_2) = \log_\alpha \left[1 + \frac{(\alpha^{u_1} - 1)(\alpha^{u_2} - 1)}{\alpha - 1}\right]$ for $\alpha > 1$; (4 marks)

(d) $C(u_1, u_2) = \left\{\left[\left(u_1^{-\theta} - 1\right)^\delta + \left(u_2^{-\theta} - 1\right)^\delta\right]^{1/\delta} + 1\right\}^{-1/\theta}$ for $\theta > 0$ and $\delta \ge 1$. (4 marks)

(Total marks: 20)

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A3. Suppose a random vector (X, Y) has the joint survival function specified by

$$\overline{G}(x,y) = \begin{cases} \exp\left\{-(x+y)A_1\left(\frac{y}{x+y}\right)\right\}, & \text{if } 0 \leq \frac{y}{x+y} \leq w_1, \\ \exp\left\{-(x+y)A_2\left(\frac{y}{x+y}\right)\right\}, & \text{if } w_1 < \frac{y}{x+y} \leq w_2, \\ \vdots & \\ \exp\left\{-(x+y)A_p\left(\frac{y}{x+y}\right)\right\}, & \text{if } w_{p-1} \leq \frac{y}{x+y} \leq 1, \end{cases}$$

where A_i, $i = 1, 2, \ldots, k$ are convex functions on $[0, 1]$ satisfying $A_i(0) = 1$, $A_i(1) = 1$ and $\max(w, 1-w) \leq A_i(w) \leq 1$ for all w and α_i, $i = 1, 2, \ldots, k$ are non-negative and sum to 1.

(a) Show that the distribution is a bivariate extreme value distribution; (7 marks)

(b) Derive the joint cumulative distribution function of X and Y; (1 marks)

(c) Derive the conditional cumulative distribution function of Y given $X = x$. You may express this in terms of $A_i'(w)$, the first derivative of $A_i(w)$; (4 marks)

(d) Derive the conditional cumulative distribution function of X given $Y = y$. You may express this in terms of $A_i'(w)$, the first derivative of $A_i(w)$; (4 marks)

(e) Derive the joint probability density function of X and Y. You may express this in terms of $A_i'(w)$ and $A_i''(w)$, the first and second derivatives of $A_i(w)$. (4 marks)

(Total marks: 20)

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SECTION B

Answer any **FOUR** questions

B1. Suppose a portfolio consists of N investments, where N is a Geometric (θ) random variable. Suppose the losses on the investments $X_1, X_2, \ldots, X_N$ are independent random variables having the cdf $[1 + \exp(-x)]^{-1}$. Suppose also $X_1, X_2, \ldots, X_N$ are independent of N . Let $T = \max(X_1, X_2, \ldots, X_N)$ denote the maximum portfolio loss.

(a) Determine the cumulative distribution function of T conditional on $N = n$; (4 marks)

(b) Hence, determine the unconditional cumulative distribution function of T; (4 marks)

(c) Hence, deduce the unconditional probability density function of T; (1 marks)

(d) Find the moment generating function of T; (4 marks)

(e) Find the value at risk of T; (3 marks)

(f) Find the expected shortfall of T. (4 marks)

(Total marks: 20)

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B2. (a) Suppose $X_1, X_2, \ldots, X_n$ is a random sample with cdf $F(\cdot)$. State the Extremal Types Theorem for $M_n = \max(X_1, X_2, \ldots, X_n)$. You must clearly specify the cdfs of each of the three extreme value distributions. (4 marks)

(b) State in full the necessary and sufficient conditions for $F(\cdot)$ to belong to the domain of attraction of each of the three extreme value distributions. (4 marks)

(c) Consider a class of distributions defined by the pdf

$$f(x) = Cg(x)\left[1 - G(x)\right]^{\lambda b - 1}\left\{1 - \left[1 - G(x)\right]^{\lambda}\right\}^{a-1}$$

where $a > 0$, $b > 0$, $\lambda > 0$, $G(\cdot)$ is a cdf, $g(x) = dG(x)/dx$ and C is a constant. Show that F belongs to the same max domain of attraction as G. You may assume that F and G have the same upper end points. (12 marks)

(Total marks: 20)

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B3. Determine the domain of attraction (if there is one) for each of the following distributions:

(a) The distribution given by the pmf

$$p(k) = \binom{n}{k} \frac{B(k+\alpha, n-k+\beta)}{B(\alpha, \beta)}$$

for $\alpha > 0$, $\beta > 0$ and $k = 1, 2, \ldots, n$; (4 marks)

(b) The distribution given by the pmf

$$p(k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

for $N \geq 1$, $0 \leq K \leq N$, $0 \leq n \leq N$ and $k = \max(0, n+K-N), \ldots, \min(n, K)$; (4 marks)

(c) The distribution given by the pdf

$$f(x) = C \exp\left[bx - \eta \exp(bx)\right]$$

for $x > 0$, $b > 0$, $\eta > 0$ and C a constant; (4 marks)

(d) The distribution given by the pdf

$$f(x) = C x^{-\alpha-1} \exp\left(-\frac{\beta}{x}\right)$$

for $x > 0$, $\alpha > 0$, $\beta > 0$ and C a constant; (4 marks)

(e) The distribution given by the cdf

$$F(x) = \left[1 + \exp\left(-ax\right)\right]^{-b}$$

for $-\infty < x < \infty$, $a > 0$ and $b > 0$. (4 marks)

(Total marks: 20)

B4. (a) If X is an absolutely continuous random variable with cdf $F(\cdot)$, then define $\mathrm{VaR}_p(X)$, the Value at Risk of X, and $\mathrm{ES}_p(X)$, the Expected Shortfall of X, explicitly. (2 marks)

(b) Suppose $X_1, X_2, \ldots, X_k$ are losses on k investments. Suppose also $X_1, X_2, \ldots, X_k$ are independent $N\left(\mu_i, \sigma_i^2\right)$, $i = 1, 2, \ldots, k$ random variables. Let $T = X_1 + X_2 + \cdots + X_k$ denote the total loss.

(i) Determine the distribution of T; (2 marks)

(ii) Derive the corresponding $\mathrm{VaR}_p(T)$; (2 marks)

(iii) Derive the corresponding $\mathrm{ES}_p(T)$. (2 marks)

(c) Suppose $X_{i,1}, X_{i,2}, \ldots, X_{i,n}$ is a random sample on X_i in (b). Suppose also $X_{i,1}, X_{i,2}, \ldots, X_{i,n}$ and $X_{j,1}, X_{j,2}, \ldots, X_{j,n}$ are independent for $i \neq j$.

(i) Write down the joint likelihood function of $\mu_1, \mu_2, \ldots, \mu_k$ and $\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2$; (2 marks)

(ii) Show that the maximum likelihood estimators are

$$\widehat{\mu_i} = \frac{1}{n} \sum_{j=1}^{n} X_{i,j}$$

and

$$\widehat{\sigma_i^2} = \frac{1}{n} \sum_{j=1}^{n} \left(X_{i,j} - \widehat{\mu_i}\right)^2;$$

(4 marks)

(iii) Show that $\widehat{\mu_i}$ is unbiased and consistent for μ_i; (2 marks)

(iv) Show that $\widehat{\sigma_i^2}$ is biased and consistent for σ_i^2; (2 marks)

(v) Deduce the maximum likelihood estimators of $\mathrm{VaR}_p(T)$ and $\mathrm{ES}_p(T)$. (2 marks)

(Total marks: 20)

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B5. Suppose a portfolio is made up of k investments where k is known. Suppose also that the losses on the investments say X_i, $i = 1, 2, \ldots, k$ are dependent random variables with joint survival function

$$\overline{F}(x_1, x_2, \ldots, x_k) = \left[\max\left(\frac{x_1}{a_1}, \frac{x_2}{a_2}, \ldots, \frac{x_k}{a_k} \right) \right]^{-a}$$

for $a > 0$ and $x_i > a_i$, $i = 1, 2, \ldots, k$, where both $a > 0$ and $a_i > 0$, $i = 1, 2, \ldots, k$ are unknown parameters.

(a) Show that the cdf of the minimum portfolio loss $T = \min(X_1, X_2, \ldots, X_k)$ is

$$F_T(t) = 1 - \left[\min(a_1, a_2, \ldots, a_k)\right]^a t^{-a};$$

(6 marks)

(b) Derive the corresponding pdf of T; (2 marks)

(c) Derive the corresponding $\mathrm{VaR}_p(T)$; (2 marks)

(d) Derive the corresponding $\mathrm{ES}_p(T)$; (2 marks)

(e) If $t_1, t_2, \ldots, t_n$ is a random sample on T derive the maximum likelihood estimates of a and $a_1, a_2, \ldots, a_k$. (8 marks)

(Total marks: 20)

END OF EXAMINATION PAPER