MATH68181

Three hours

To be supplied by the Examinations Office: Mathematical Formula Tables and Statistical Tables

THE UNIVERSITY OF MANCHESTER

EXTREME VALUES AND FINANCIAL RISK

Examiner:

20 January 2017

9:45am-12:45pm

Answer ANY TWO questions in Section A.
Answer ANY FOUR questions in Section B.

Electronic calculators are permitted provided that cannot store text.

SECTION A

Answer any **TWO** questions

A1. Suppose (X, Y) has the joint cdf specified by

$$F_{X,Y}(x, y) = 1 - \exp(-x^\alpha) - \exp(-y^\alpha) + [\exp(x^\alpha) + \exp(y^\alpha) - 1]^{-1}$$

for $x > 0$, $y > 0$ and $\alpha > 0$.

(a) Find the marginal cdfs of X and Y, that is $F_X(\cdot)$ and $F_Y(\cdot)$; (3 marks)

(b) Show that F_X belongs to the Gumbel max domain of attraction; (3 marks)

(c) Show that F_Y also belongs to the Gumbel max domain of attraction; (2 marks)

(d) Find a_n and b_n such that

$$F_X^n(a_n x + b_n) \to \exp\{-\exp(-x)\}$$

as $n \to \infty$; (3 marks)

(e) Find c_n and d_n such that

$$F_Y^n(c_n x + d_n) \to \exp\{-\exp(-x)\}$$

as $n \to \infty$; (2 marks)

(f) Find the limiting cdf of $F_{X,Y}^n(a_n x + b_n, c_n y + d_n)$ as $n \to \infty$; (5 marks)

(g) Are the extremes of (X, Y) completely independent? Justify your answer. (2 marks)

(Total marks: 20)

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A2. State the conditions in full for $C(u_1, u_2)$, $0 \leq u_1, u_2 \leq 1$ to be a copula. (4 marks)

Show each of the following is a copula.

(a) The copula defined by $C(u_1, u_2) = \alpha C_1(u_1, u_2) + (1-\alpha)C_2(u_1, u_2)$ for $0 < \alpha < 1$, where C_1 and C_2 are valid copulas. (4 marks)

(b) The copula defined by $C(u_1, u_2) = \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2}$. (4 marks)

(c) The copula defined by $C(u_1, u_2) = u_1 u_2 + \alpha u_1 u_2 (1-u_1)(1-u_2)$ for $-1 < \alpha < 1$. (4 marks)

(d) The copula defined by $C(u_1, u_2) = \begin{cases} \max(u_1 + u_2 - 1, t), & \text{if } t \leq u_1 \leq 1, t \leq u_2 \leq 1, \\ \min(u_1, u_2), & \text{otherwise} \end{cases}$ for $0 < t < 1$. (4 marks)

(Total marks: 20)

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A3. Consider a bivariate distribution specified by the joint survival function

$$\overline{G}(x, y) = \exp\left\{-(x+y)\sum_{i=1}^{k} \alpha_i A_i\left(\frac{y}{x+y}\right)\right\}$$

for $x > 0$ and $y > 0$, where A_i, $i = 1, 2, \ldots, k$ are convex functions on $[0, 1]$ satisfying $A_i(0) = 1$, $A_i(1) = 1$ and $\max(w, 1-w) \leq A_i(w) \leq 1$ for all w and α_i, $i = 1, 2, \ldots, k$ are non-negative and sum to 1.

(a) Show that the distribution is a bivariate extreme value distribution; (7 marks)

(b) Derive the joint cumulative distribution function; (1 marks)

(c) Derive the conditional cumulative distribution function of Y given $X = x$. You may express this in terms of $A_i^{'}(w)$, the first derivative of $A_i(w)$; (4 marks)

(d) Derive the conditional cumulative distribution function of X given $Y = y$. You may express this in terms of $A_i^{'}(w)$, the first derivative of $A_i(w)$; (4 marks)

(e) Derive the joint probability density function. You may express this in terms of $A_i^{'}(w)$ and $A_i^{''}(w)$, the first and second derivatives of $A_i(w)$. (4 marks)

(Total marks: 20)

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SECTION B

Answer any **FOUR** questions

B1. Suppose a portfolio consists of N investments, where N is a Poisson (θ) random variable. Suppose the losses on the investments $X_1, X_2, \ldots, X_N$ are independent uniform $[-a, a]$ random variables independent of N. Let $T = \max(X_1, X_2, \ldots, X_N)$ denote the maximum portfolio loss.

(a) Determine the cumulative distribution function of T conditional on $N = n$; (4 marks)

(b) Hence, determine the unconditional cumulative distribution function of T; (4 marks)

(c) Hence, deduce the unconditional probability density function of T; (1 marks)

(d) Find the moment generating function of T; (4 marks)

(e) Find the value at risk of T; (3 marks)

(f) Find the expected shortfall of T. (4 marks)

(Total marks: 20)

B2. (a) Suppose $X_1, X_2, \ldots, X_n$ is a random sample with cdf $F(\cdot)$. State the Extremal Types Theorem for $M_n = \max(X_1, X_2, \ldots, X_n)$. You must clearly specify the cdfs of each of the three extreme value distributions. (4 marks)

(b) State in full the necessary and sufficient conditions for $F(\cdot)$ to belong to the domain of attraction of each of the three extreme value distributions. (4 marks)

(c) Consider a class of distributions defined by the pdf

$$f(x) = Cg(x)G^{a-1}(x)\left[1 - G(x)\right]^{b-1}$$

where $a > 0$, $b > 0$, $G(\cdot)$ is a cdf, $g(x) = dG(x)/dx$ and C is a constant. Show that F belongs to the same max domain of attraction as G. You may assume that F and G have the same upper end points. (12 marks)

(Total marks: 20)

B3. Determine the domain of attraction (if there is one) for each of the following distributions:

(a) The distribution given by the pmf

$$p(k) = \frac{1}{N}$$

for $k = 1, 2, \ldots, N$; (4 marks)

(b) The distribution given by the pmf

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for $n \geq 1$, $0 < p < 1$ and $k = 0, 1, \ldots, n$; (4 marks)

(c) The distribution given by the pdf

$$f(x) = \frac{1}{(\log b - \log a)\, x}, 0 < a < x < b < \infty;$$

(4 marks)

(d) The distribution given by the cdf

$$F(x) = \left[1 - \exp\left(-x^2\right)\right]^a, a > 0, x > 0;$$

(4 marks)

(e) The distribution given by the cdf

$$F(x) = 1 - \left[1 - \exp\left(-x^{-1}\right)\right]^a, a > 0, x > 0.$$

(4 marks)

(Total marks: 20)

B4. (a) If X is an absolutely continuous random variable with cdf $F(\cdot)$, then define $\mathrm{VaR}_p(X)$, the Value at Risk of X, and $\mathrm{ES}_p(X)$, the Expected Shortfall of X, explicitly. (2 marks)

(b) Suppose X is a random variable with pdf given by

$$f(x) = \frac{3x^2}{2a^3}$$

for $-a < x < a$.

(i) Show that the corresponding cdf is

$$F(x) = \frac{x^3 + a^3}{2a^3}$$

for $-a < x < a$; (2 marks)

(ii) Derive the corresponding $\mathrm{VaR}_p(X)$; (1 marks)

(iii) Derive the corresponding $\mathrm{ES}_p(X)$. (2 marks)

(c) Suppose $X_1, X_2, \ldots, X_n$ is a random sample on X in (b).

(i) Write down the likelihood function of a; (3 marks)

(ii) Show that the maximum likelihood estimator of a is

$$\widehat{a} = \max \left[\max \left(X_1, X_2, \ldots, X_n \right), - \min \left(X_1, X_2, \ldots, X_n \right) \right];$$

(1 marks)

(iii) Deduce the maximum likelihood estimators of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$; (2 marks)

(iv) Let $Z = \max \left[\max \left(X_1, X_2, \ldots, X_n \right), - \min \left(X_1, X_2, \ldots, X_n \right) \right]$. Show that the cdf of Z is

$$F_Z(z) = \left[\frac{z^3}{a^3} \right]^n$$

for $z > 0$; (3 marks)

(v) Hence, show that the maximum likelihood estimator $\widehat{a}$ is biased but consistent for a; (2 marks)

(vi) Hence, show that the maximum likelihood estimator of $\mathrm{VaR}_p(X)$ is also biased but consistent for $\mathrm{VaR}_p(X)$; (1 marks)

(vii) Hence, show that the maximum likelihood estimator of $\mathrm{ES}_p(X)$ is also biased but consistent for $\mathrm{ES}_p(X)$. (1 marks)

(Total marks: 20)

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B5. Suppose a portfolio is made up of k investments where k is known. Suppose also that the losses on the investments say X_i, $i = 1, 2, \ldots, k$ are dependent random variables with joint survival function

$$\overline{F}\left(x_1, x_2, \ldots, x_k\right) = \left(1 + \frac{1}{\theta} \sum_{i=1}^{k} x_i\right)^{-\alpha}$$

for $x_i > 0$, $i = 1, 2, \ldots, k$, where both $\theta > 0$ and $\alpha > 0$ are unknown parameters.

(a) Show that the joint pdf of $X_1, X_2, \ldots, X_k$ is

$$f\left(x_1, x_2, \ldots, x_k\right) = \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\theta^k} \left(1 + \frac{1}{\theta} \sum_{i=1}^{k} x_i\right)^{-\alpha-k};$$

(4 marks)

(b) Show that the pdf of the total portfolio loss $S = X_1 + \cdots + X_k$ is

$$f_S(s) = \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!\theta^k} s^k \left(1 + \frac{s}{\theta}\right)^{-\alpha-k}.$$

You may use the following fact without proof:

$$\int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{k-1}} dx_k \cdots dx_2 dx_1 = \frac{s^k}{k!};$$

(4 marks)

(c) Show that the cdf of the total portfolio loss $S = X_1 + \cdots + X_k$ is

$$F_S(s) = \frac{\theta\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!} B_{s/(s+\theta)}(k+1, \alpha-1),$$

where

$$B_x(a, b) = \int_0^x t^{a-1}(1-t)^{b-1} dt$$

denotes the incomplete beta function; (4 marks)

(d) Determine the nth moment of S; (4 marks)

(e) If $s_1, s_2, \ldots, s_n$ is a random sample on S derive the maximum likelihood estimates of θ and α. (4 marks)

(Total marks: 20)

END OF EXAMINATION PAPER