Two hours

To be supplied by the Examinations Office: Mathematical Formula Tables and Statistical Tables

THE UNIVERSITY OF MANCHESTER

EXTREME VALUES AND FINANCIAL RISK

Examiner:

20 January 2017

9:45am-11:45am

Answer any FOUR questions.

Electronic calculators are permitted provided that cannot store text.

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1. Suppose a portfolio consists of N investments, where N is a Poisson (θ) random variable. Suppose the losses on the investments $X_1, X_2, \ldots, X_N$ are independent uniform $[-a, a]$ random variables independent of N. Let $T = \max(X_1, X_2, \ldots, X_N)$ denote the maximum portfolio loss.

(a) Determine the cumulative distribution function of T conditional on $N = n$; (4 marks)

(b) Hence, determine the unconditional cumulative distribution function of T; (4 marks)

(c) Hence, deduce the unconditional probability density function of T; (1 marks)

(d) Find the moment generating function of T; (4 marks)

(e) Find the value at risk of T; (3 marks)

(f) Find the expected shortfall of T. (4 marks)

(Total marks: 20)

2. (a) Suppose $X_1, X_2, \ldots, X_n$ is a random sample with cdf $F(\cdot)$. State the Extremal Types Theorem for $M_n = \max\left(X_1, X_2, \ldots, X_n\right)$. You must clearly specify the cdfs of each of the three extreme value distributions. (4 marks)

(b) State in full the necessary and sufficient conditions for $F(\cdot)$ to belong to the domain of attraction of each of the three extreme value distributions. (4 marks)

(c) Consider a class of distributions defined by the pdf

$$f(x) = Cg(x)G^{a-1}(x)\left[1 - G(x)\right]^{b-1}$$

where $a > 0$, $b > 0$, $G(\cdot)$ is a cdf, $g(x) = dG(x)/dx$ and C is a constant. Show that F belongs to the same max domain of attraction as G. You may assume that F and G have the same upper end points. (12 marks)

(Total marks: 20)

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3. Determine the domain of attraction (if there is one) for each of the following distributions:

(a) The distribution given by the pmf

$$p(k) = \frac{1}{N}$$

for $k = 1, 2, \ldots, N$; (4 marks)

(b) The distribution given by the pmf

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for $n \geq 1$, $0 < p < 1$ and $k = 0, 1, \ldots, n$; (4 marks)

(c) The distribution given by the pdf

$$f(x) = \frac{1}{(\log b - \log a)\, x}, 0 < a < x < b < \infty;$$

(4 marks)

(d) The distribution given by the cdf

$$F(x) = \left[1 - \exp\left(-x^2\right)\right]^a, a > 0, x > 0;$$

(4 marks)

(e) The distribution given by the cdf

$$F(x) = 1 - \left[1 - \exp\left(-x^{-1}\right)\right]^a, a > 0, x > 0.$$

(4 marks)

(Total marks: 20)

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4. (a) If X is an absolutely continuous random variable with cdf $F(\cdot)$, then define $\mathrm{VaR}_p(X)$, the Value at Risk of X, and $\mathrm{ES}_p(X)$, the Expected Shortfall of X, explicitly. (2 marks)

(b) Suppose X is a random variable with pdf given by

$$f(x) = \frac{3x^2}{2a^3}$$

for $-a < x < a$.

(i) Show that the corresponding cdf is

$$F(x) = \frac{x^3 + a^3}{2a^3}$$

for $-a < x < a$; (2 marks)

(ii) Derive the corresponding $\mathrm{VaR}_p(X)$; (1 marks)

(iii) Derive the corresponding $\mathrm{ES}_p(X)$. (2 marks)

(c) Suppose $X_1, X_2, \ldots, X_n$ is a random sample on X in (b).

(i) Write down the likelihood function of a; (3 marks)

(ii) Show that the maximum likelihood estimator of a is

$$\widehat{a} = \max \left[\max \left(X_1, X_2, \ldots, X_n \right), - \min \left(X_1, X_2, \ldots, X_n \right) \right];$$

(1 marks)

(iii) Deduce the maximum likelihood estimators of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$; (2 marks)

(iv) Let $Z = \max \left[\max \left(X_1, X_2, \ldots, X_n \right), - \min \left(X_1, X_2, \ldots, X_n \right) \right]$. Show that the cdf of Z is

$$F_Z(z) = \left[\frac{z^3}{a^3} \right]^n$$

for $z > 0$; (3 marks)

(v) Hence, show that the maximum likelihood estimator $\widehat{a}$ is biased but consistent for a; (2 marks)

(vi) Hence, show that the maximum likelihood estimator of $\mathrm{VaR}_p(X)$ is also biased but consistent for $\mathrm{VaR}_p(X)$; (1 marks)

(vii) Hence, show that the maximum likelihood estimator of $\mathrm{ES}_p(X)$ is also biased but consistent for $\mathrm{ES}_p(X)$. (1 marks)

(Total marks: 20)

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5. Suppose a portfolio is made up of k investments where k is known. Suppose also that the losses on the investments say X_i, $i = 1, 2, \ldots, k$ are dependent random variables with joint survival function

$$\overline{F}\left(x_1, x_2, \ldots, x_k\right) = \left(1 + \frac{1}{\theta} \sum_{i=1}^{k} x_i\right)^{-\alpha}$$

for $x_i > 0$, $i = 1, 2, \ldots, k$, where both $\theta > 0$ and $\alpha > 0$ are unknown parameters.

(a) Show that the joint pdf of $X_1, X_2, \ldots, X_k$ is

$$f\left(x_1, x_2, \ldots, x_k\right) = \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\theta^k} \left(1 + \frac{1}{\theta} \sum_{i=1}^{k} x_i\right)^{-\alpha-k};$$

(4 marks)

(b) Show that the pdf of the total portfolio loss $S = X_1 + \cdots + X_k$ is

$$f_S(s) = \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!\theta^k} s^k \left(1 + \frac{s}{\theta}\right)^{-\alpha-k}.$$

You may use the following fact without proof:

$$\int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{k-1}} dx_k \cdots dx_2 dx_1 = \frac{s^k}{k!};$$

(4 marks)

(c) Show that the cdf of the total portfolio loss $S = X_1 + \cdots + X_k$ is

$$F_S(s) = \frac{\theta\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!} B_{s/(s+\theta)}(k+1, \alpha-1),$$

where

$$B_x(a, b) = \int_0^x t^{a-1}(1-t)^{b-1} dt$$

denotes the incomplete beta function; (4 marks)

(d) Determine the nth moment of S; (4 marks)

(e) If $s_1, s_2, \ldots, s_n$ is a random sample on S derive the maximum likelihood estimates of θ and α. (4 marks)

(Total marks: 20)

END OF EXAMINATION PAPER