Three hours

To be supplied by the Examinations Office: Mathematical Formula Tables and Statistical Tables

THE UNIVERSITY OF MANCHESTER

EXTREME VALUES AND FINANCIAL RISK

Examiner:

19 January 2016

14:00pm-17:00pm

Answer ANY TWO questions in Section A.
Answer ANY FOUR questions in Section B.

Electronic calculators are permitted provided that cannot store text.

SECTION A

Answer any **TWO** questions

A1. Suppose (X, Y) has the joint cdf specified by

$$F_{X,Y}(x, y) = [1 + \exp(-x) + \exp(-y) + (1 - \alpha)\exp(-x - y)]^{-1}$$

for $-\infty < x < \infty$, $-\infty < y < \infty$ and $0 < \alpha < 1$.

(a) Find the marginal cdfs of X and Y, that is $F_X(\cdot)$ and $F_Y(\cdot)$; (3 marks)

(b) Show that F_X belongs to the Gumbel max domain of attraction; (3 marks)

(c) Show that F_Y also belongs to the Gumbel max domain of attraction; (2 marks)

(d) Find a_n and b_n such that

$$F_X^n(a_n x + b_n) \to \exp\{-\exp(-x)\}$$

as $n \to \infty$; (3 marks)

(e) Find c_n and d_n such that

$$F_Y^n(c_n x + d_n) \to \exp\{-\exp(-x)\}$$

as $n \to \infty$; (2 marks)

(f) Find the limiting cdf of $F_{X,Y}^n(a_n x + b_n, c_n y + d_n)$ as $n \to \infty$; (5 marks)

(g) Are the extremes of (X, Y) completely independent? Justify your answer. (2 marks)

(Total marks: 20)

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A2. State the conditions in full for $C(u_1, u_2)$, $0 \leq u_1, u_2 \leq 1$ to be a copula. (4 marks)

Show each of the following is a copula.

(a) The copula defined by $C(u_1, u_2) = \left[\alpha \left(\min\left(u_1, u_2\right)\right)^m + (1-\alpha) u_1^m u_2^m\right]^{1/m}$ for $0 < \alpha < 1$ and $m > 0$. (4 marks)

(b) The copula defined by $C(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$. (4 marks)

(c) The copula defined by $C(u_1, u_2) = \min\left(u_1^a, u_2^b\right) \min\left(u_1^{1-a}, u_2^{1-b}\right)$ for $0 < a, b < 1$. (4 marks)

(d) The copula defined by $C(u_1, u_2) = \exp\left\{-\left[\left(-\log u_1\right)^\theta + \left(-\log u_2\right)^\theta\right]^{1/\theta}\right\}$ for $\theta > 0$. (4 marks)

(Total marks: 20)

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A3. Consider a bivariate distribution specified by the joint survival function

$$\overline{G}(x,y) = \exp\left\{-x - y + (\theta+\phi)y - \frac{\theta y^2}{x+y} - \frac{\phi y^3}{(x+y)^2}\right\}$$

for $x > 0$, $y > 0$, $\theta \geq 0$, $\phi \geq 0$, $\theta + 3\phi \geq 0$, $\theta + \phi \leq 1$ and $\theta + 2\phi \leq 1$.

(a) Show that the distribution is a bivariate extreme value distribution; (6 marks)

(b) Derive the joint cumulative distribution function; (2 marks)

(c) Derive the conditional cumulative distribution function of Y given $X = x$; (4 marks)

(d) Derive the conditional cumulative distribution function of X given $Y = y$; (4 marks)

(e) Derive the joint probability density function. (4 marks)

(Total marks: 20)

SECTION B

Answer any **FOUR** questions

B1. Suppose a portfolio consists of N investments, where N is a Geometric (θ) random variable. Suppose the losses on the investments $X_1, X_2, \ldots, X_N$ are independent and identical Exponential (λ) random variables independent of N. Let $T = X_1 + X_2 + \cdots + X_N$ denote the total portfolio loss.

(a) Show that the moment generating function of X_i is

$$M_{X_i}(t) = \frac{\lambda}{\lambda - t}$$

for $t < \lambda$; (2 marks)

(b) Deduce the moment generating function of $T = X_1 + X_2 + \cdots + X_N$ conditional on $N = n$; (3 marks)

(c) Hence, determine the distribution of $T = X_1 + X_2 + \cdots + X_N$ conditional on $N = n$; (3 marks)

(d) Determine the unconditional distribution of $T = X_1 + X_2 + \cdots + X_N$; (3 marks)

(e) Find the mean and variance of T; (3 marks)

(f) Find the value at risk of T; (3 marks)

(g) Find the expected shortfall of T. (3 marks)

(Total marks: 20)

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B2. (a) Suppose $X_1, X_2, \ldots, X_n$ is a random sample with cdf $F(\cdot)$. State the Extremal Types Theorem for $M_n = \max(X_1, X_2, \ldots, X_n)$. You must clearly specify the cdfs of each of the three extreme value distributions. (6 marks)

(b) State in full the necessary and sufficient conditions for $F(\cdot)$ to belong to the domain of attraction of each of the three extreme value distributions. (6 marks)

(c) Consider a class of distributions defined by the cdf

$$F(x) = 1 - \left\{ 1 - \{1 - [1 - G(x)]^a\}^b \right\}^\theta,$$

where $a > 0$, $b > 0$, $\theta > 0$, $G(\cdot)$ is a cdf and $g(x) = dG(x)/dx$. Show that F belongs to the same max domain of attraction as G. You may assume that F and G have the same upper end points. (8 marks)

(Total marks: 20)

B3. Determine the domain of attraction (if there is one) for each of the following distributions:

(a) The beta distribution given by the pdf

$$f(x) = Cx^{\alpha-1}(1-x)^{\beta-1}, 0 < x < 1, \alpha > 0, \beta > 0,$$

where C is a fixed constant; (4 marks)

(b) The distribution given by the pmf

$$p(k) = \begin{cases} 1/2, & \text{if } k = -1, 1, \\ 0, & \text{otherwise;} \end{cases}$$

(4 marks)

(c) The Cauchy distribution given by the pdf

$$f(x) = \pi^{-1}\left(1+x^2\right)^{-1}, -\infty < x < \infty;$$

(4 marks)

(d) The Laplace distribution given by the pdf

$$f(x) = 0.5e^{-|x|}, -\infty < x < \infty;$$

(4 marks)

(e) The Fréchet distribution given by the cdf

$$F(x) = \exp\left\{-x^{-1}\right\}, x > 0.$$

(4 marks)

(Total marks: 20)

B4. (a) If X is an absolutely continuous random variable with cdf $F(\cdot)$, then define $\mathrm{VaR}_p(X)$, the Value at Risk of X, and $\mathrm{ES}_p(X)$, the Expected Shortfall of X, explicitly. (2 marks)

(b) Suppose X is a Pareto random variable with pdf given by

$$f(x) = aK^a x^{-a-1}$$

for $a > 0$, $K > 0$ and $x > K$.

(i) Show that the corresponding cdf is

$$F(x) = 1 - \left(\frac{K}{x}\right)^a$$

for $x > K$; (2 marks)

(ii) Derive the corresponding $\mathrm{VaR}_p(X)$; (2 marks)

(iii) Derive the corresponding $\mathrm{ES}_p(X)$. (2 marks)

(c) Suppose $X_1, X_2, \ldots, X_n$ is a random sample on X.

(i) Write down the joint likelihood function of a and K; (1 marks)

(ii) Show that the maximum likelihood estimator of K is $\widehat{K} = \min(X_1, X_2, \ldots, X_n)$; (2 marks)

(iii) Show that the maximum likelihood estimator of a is $\widehat{a} = n\left[-n\log\widehat{K} + \sum_{i=1}^{n}\log X_i\right]^{-1}$; (2 marks)

(iv) Deduce the maximum likelihood estimators of $\mathrm{VaR}_0(X)$ and $\mathrm{ES}_0(X)$; (2 marks)

(v) Show that the maximum likelihood estimators in part iv) are biased [Hint: derive the distribution of $\min(X_1, X_2, \ldots, X_n)$]. (5 marks)

(Total marks: 20)

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B5. Suppose a portfolio is made up of m investments where m is known. Suppose also that the losses on the investments say X_i, $i = 1, 2, \ldots, m$ are independent and identical uniform $[a, b]$ random variables, where both a and b are unknown parameters. Let $Y = \max(X_1, \ldots, X_m)$. Do the following:

(a) Show that the cdf of Y is

$$F_Y(y) = \left[\frac{y-a}{b-a}\right]^m;$$

(2 marks)

(b) Find the pdf of Y; (2 marks)

(c) Find the mean and variance of Y; (4 marks)

(d) Find the value at risk of Y; (2 marks)

(e) Find the expected shortfall of Y; (4 marks)

(f) If $y_1, y_2, \ldots, y_n$ is a random sample on Y derive the maximum likelihood estimates of a and b. (6 marks)

(Total marks: 20)

END OF EXAMINATION PAPER