MATH38181

Two hours

To be supplied by the Examinations Office: Mathematical Formula Tables and Statistical Tables

THE UNIVERSITY OF MANCHESTER

EXTREME VALUES AND FINANCIAL RISK

Examiner:

19 January 2016

14:00pm-16:00pm

Answer any FOUR questions.

Electronic calculators are permitted provided that cannot store text.

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1. Suppose a portfolio consists of N investments, where N is a Geometric (θ) random variable. Suppose the losses on the investments $X_1, X_2, \ldots, X_N$ are independent and identical Exponential (λ) random variables independent of N. Let $T = X_1 + X_2 + \cdots + X_N$ denote the total portfolio loss.

(a) Show that the moment generating function of X_i is

$$M_{X_i}(t) = \frac{\lambda}{\lambda - t}$$

for $t < \lambda$; (2 marks)

(b) Deduce the moment generating function of $T = X_1 + X_2 + \cdots + X_N$ conditional on $N = n$; (3 marks)

(c) Hence, determine the distribution of $T = X_1 + X_2 + \cdots + X_N$ conditional on $N = n$; (3 marks)

(d) Determine the unconditional distribution of $T = X_1 + X_2 + \cdots + X_N$; (3 marks)

(e) Find the mean and variance of T; (3 marks)

(f) Find the value at risk of T; (3 marks)

(g) Find the expected shortfall of T. (3 marks)

(Total marks: 20)

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2. (a) Suppose $X_1, X_2, \ldots, X_n$ is a random sample with cdf $F(\cdot)$. State the Extremal Types Theorem for $M_n = \max(X_1, X_2, \ldots, X_n)$. You must clearly specify the cdfs of each of the three extreme value distributions. (6 marks)

(b) State in full the necessary and sufficient conditions for $F(\cdot)$ to belong to the domain of attraction of each of the three extreme value distributions. (6 marks)

(c) Consider a class of distributions defined by the cdf

$$F(x) = 1 - \left\{1 - \{1 - [1 - G(x)]^a\}^b\right\}^\theta,$$

where $a > 0$, $b > 0$, $\theta > 0$, $G(\cdot)$ is a cdf and $g(x) = dG(x)/dx$. Show that F belongs to the same max domain of attraction as G. You may assume that F and G have the same upper end points. (8 marks)

(Total marks: 20)

3. Determine the domain of attraction (if there is one) for each of the following distributions:

(a) The beta distribution given by the pdf

$$f(x) = Cx^{\alpha-1}(1-x)^{\beta-1}, 0 < x < 1, \alpha > 0, \beta > 0,$$

where C is a fixed constant; (4 marks)

(b) The distribution given by the pmf

$$p(k) = \begin{cases} 1/2, & \text{if } k = -1, 1, \\ 0, & \text{otherwise;} \end{cases}$$

(4 marks)

(c) The Cauchy distribution given by the pdf

$$f(x) = \pi^{-1}\left(1 + x^2\right)^{-1}, -\infty < x < \infty;$$

(4 marks)

(d) The Laplace distribution given by the pdf

$$f(x) = 0.5e^{-|x|}, -\infty < x < \infty;$$

(4 marks)

(e) The Fréchet distribution given by the cdf

$$F(x) = \exp\left\{-x^{-1}\right\}, x > 0.$$

(4 marks)

(Total marks: 20)

4. (a) If X is an absolutely continuous random variable with cdf $F(\cdot)$, then define $\mathrm{VaR}_p(X)$, the Value at Risk of X, and $\mathrm{ES}_p(X)$, the Expected Shortfall of X, explicitly. (2 marks)

(b) Suppose X is a Pareto random variable with pdf given by

$$f(x) = aK^a x^{-a-1}$$

for $a > 0$, $K > 0$ and $x > K$.

(i) Show that the corresponding cdf is

$$F(x) = 1 - \left(\frac{K}{x}\right)^a$$

for $x > K$; (2 marks)

(ii) Derive the corresponding $\mathrm{VaR}_p(X)$; (2 marks)

(iii) Derive the corresponding $\mathrm{ES}_p(X)$. (2 marks)

(c) Suppose $X_1, X_2, \ldots, X_n$ is a random sample on X.

(i) Write down the joint likelihood function of a and K; (1 marks)

(ii) Show that the maximum likelihood estimator of K is $\widehat{K} = \min\left(X_1, X_2, \ldots, X_n\right)$; (2 marks)

(iii) Show that the maximum likelihood estimator of a is $\widehat{a} = n\left[-n\log\widehat{K} + \sum_{i=1}^{n}\log X_i\right]^{-1}$; (2 marks)

(iv) Deduce the maximum likelihood estimators of $\mathrm{VaR}_0(X)$ and $\mathrm{ES}_0(X)$; (2 marks)

(v) Show that the maximum likelihood estimators in part iv) are biased [Hint: derive the distribution of $\min\left(X_1, X_2, \ldots, X_n\right)$]. (5 marks)

(Total marks: 20)

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5. Suppose a portfolio is made up of m investments where m is known. Suppose also that the losses on the investments say X_i, $i = 1, 2, \ldots, m$ are independent and identical uniform $[a, b]$ random variables, where both a and b are unknown parameters. Let $Y = \max(X_1, \ldots, X_m)$. Do the following:

(a) Show that the cdf of Y is

$$F_Y(y) = \left[\frac{y-a}{b-a} \right]^m;$$

(2 marks)

(b) Find the pdf of Y; (2 marks)

(c) Find the mean and variance of Y; (4 marks)

(d) Find the value at risk of Y; (2 marks)

(e) Find the expected shortfall of Y; (4 marks)

(f) If $y_1, y_2, \ldots, y_n$ is a random sample on Y derive the maximum likelihood estimates of a and b. (6 marks)

(Total marks: 20)

END OF EXAMINATION PAPER