MATH68181

Three hours

To be supplied by the Examinations Office: Mathematical Formula Tables and Statistical Tables

THE UNIVERSITY OF MANCHESTER

EXTREME VALUES AND FINANCIAL RISK

Examiner:

19 August 2015

9:45am-12:45pm

Answer QUESTION 1, QUESTION 2 and any FOUR of the remaining questions.

Electronic calculators are permitted provided that cannot store text.

Turn to next page

1. Let X_i denote a normal random variable with zero mean and variance σ^2, representing the change in stock value from day i to the next.

(a) Show that the moment generating function of X_i is

$$M_{X_i}(t) = E\left[\exp\left(tX_i\right)\right] = \exp\left(\frac{\sigma^2 t^2}{2}\right).$$

(4 marks)

(b) If X_i, $i = 1, 2, \ldots, n$ are independent normal random variables with zero means and common variance σ^2 calculate the moment generating function of $Y = X_1 + \cdots + X_n$. (4 marks)

(c) Determine the first four moments of Y. (4 marks)

(d) What is the distribution of Y? (4 marks)

(e) Determine $\mathrm{VaR}_p(Y)$ and $\mathrm{ES}_p(Y)$. (4 marks)

2. State the conditions in full for $C(u_1, u_2)$, $0 \leq u_1, u_2 \leq 1$ to be a copula. (4 marks)

Show each of the following is a copula.

(a) The copula defined by $C(u_1, u_2) = \left[u_1^{-\alpha} + u_2^{-\alpha} - 1\right]^{-1/\alpha}$ for $\alpha \geq 0$. (4 marks)

(b) The copula defined by $C(u_1, u_2) = [\min(u_1, u_2)]^{\theta} (u_1 u_2)^{1-\theta}$ for $0 \leq \theta \leq 1$. (4 marks)

(c) The copula defined by $C(u_1, u_2) = \frac{u_1 u_2}{1-\theta(1-u_1)(1-u_2)}$ for $-1 \leq \theta < 1$. (4 marks)

(d) The copula defined by $(1-\rho)u_1 u_2 + \rho \min(u_1, u_2)$ for $0 < \rho < 1$. (4 marks)

(Total marks: 20)

3. Consider a bivariate distribution specified by the joint survival function

$$\overline{F}(x, y) = \exp\left\{-x - \frac{y}{3} - \frac{y^2}{3(x+y)} - \frac{y^3}{3(x+y)^2}\right\}$$

for $x > 0$ and $y > 0$;

(a) Show that the distribution is a bivariate extreme value distribution; (6 marks)

(b) Derive the joint cumulative distribution function; (2 marks)

(c) Derive the conditional cumulative distribution function of Y given $X = x$; (4 marks)

(d) Derive the conditional cumulative distribution function of X given $Y = y$; (4 marks)

(e) Derive the joint probability density function. (4 marks)

(Total marks: 20)

Turn to next page

4. Let X denote a random variable representing the stock return of a company. Suppose X has the uniform $[-\theta, \theta]$ distribution with random θ.

(a) Suppose θ follows the Fréchet distribution specified by the pdf $\lambda\theta^{-2}\exp(-\lambda/\theta)$, $\theta > 0$, where λ is an unknown parameter. Show that the unconditional cdf of X in this case is

$$F_X(x) = \frac{x + \lambda}{2\lambda}$$

for $-\lambda < x < \lambda$; (4 marks)

(b) Determine the corresponding pdf of X; (2 marks)

(c) Derive the mean of X; (4 marks)

(d) Derive the variance of X; (4 marks)

(e) If $x_1, x_2, \ldots, x_n$ is a random sample on X find the maximum likelihood estimate of λ. (6 marks)

(Total marks: 20)

5. (a) Suppose $X_1, X_2, \ldots, X_n$ is a random sample with cdf $F(\cdot)$. State the Extremal Types Theorem for $M_n = \max(X_1, X_2, \ldots, X_n)$. You must clearly specify the cdfs of each of the three extreme value distributions. (6 marks)

(b) State in full the necessary and sufficient conditions for $F(\cdot)$ to belong to the domain of attraction of each of the three extreme value distributions. (6 marks)

(c) Consider a class of distributions defined by the cdf

$$F(x) = \left\{1 - \left[1 - (G(x))^\theta\right]^2\right\}^\alpha$$

where $\alpha > 0$, $\theta > 0$, $G(\cdot)$ is a cdf and $g(x) = dG(x)/dx$. Show that F belongs to the same max domain of attraction as G. You may assume that F and G have the same upper end points. (8 marks)

(Total marks: 20)

6. Determine the domain of attraction (if there is one) for each of the following distributions:

(a) The Burr distribution given by the cdf

$$F(x) = 1 - \left(1 + x^c\right)^{-k}, x > 0, c > 0, k > 0;$$

(4 marks)

(b) The Kumaraswamy distribution given by the cdf

$$F(x) = 1 - \left(1 - x^b\right)^a, 0 < x < 1, a > 0, b > 0;$$

(4 marks)

(c) The Poisson distribution given by the pmf

$$p(k) = \frac{\lambda^k \exp(-\lambda)}{k!}, \lambda > 0, k = 0, 1, \ldots;$$

(4 marks)

(d) The distribution given by the cdf

$$F(x) = \Phi(x), -\infty < x < \infty,$$

where $\Phi(\cdot)$ denotes the cdf of a standard normal random variable; (4 marks)

(e) The Gumbel distribution given by the cdf

$$F(x) = \exp\left\{-\exp(-x)\right\}, -\infty < x < \infty.$$

(4 marks)

(Total marks: 20)

Turn to next page

7. (a) If X is an absolutely continuous random variable with cdf $F(\cdot)$, then define $\mathrm{VaR}_p(X)$, the Value at Risk of X, and $\mathrm{ES}_p(X)$, the Expected Shortfall of X, explicitly. (2 marks)

(b) Suppose X is a Laplace random variable with pdf given by

$$f(x) = 1/(2\lambda) \exp\left(- \mid x \mid / \lambda\right)$$

for $\lambda > 0$ and $-\infty < x < \infty$.

(i) Show that the corresponding cdf is

$$F(x) = \begin{cases} 1 - \dfrac{1}{2} \exp\left(-\dfrac{x}{\lambda}\right), & \text{if } x > 0, \\ \dfrac{1}{2} \exp\left(\dfrac{x}{\lambda}\right), & \text{if } x \leq 0; \end{cases}$$

(3 marks)

(ii) Derive the corresponding $\mathrm{VaR}_p(X)$; (2 marks)

(iii) Derive the corresponding $\mathrm{ES}_p(X)$. (3 marks)

(c) Suppose $X_1, X_2, \ldots, X_n$ is a random sample on X.

(i) Write down the likelihood function of λ; (2 marks)

(ii) Show that the maximum likelihood estimator (mle) of λ is $\frac{1}{n} \sum_{i=1}^{n} |X_i|$; (3 marks)

(iii) Deduce the mles of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$; (2 marks)

(iv) Show that the mles in part iii) are unbiased. (3 marks)

(Total marks: 20)

8. Suppose a portfolio is made up of α investments where α is known. Suppose also that the losses on the investments say X_i, $i = 1, 2, \ldots, \alpha$ are independent and identical Pareto random variables specified by the cdf $F(x) = 1 - (K/x)^a$, $x \geq K$, where both $a > 0$ and $K > 0$ are unknown parameters. Let $Y = \min(X_1, \ldots, X_\alpha)$. Do the following:

(a) Show that the cdf of Y is

$$F_Y(y) = 1 - \left(\frac{K}{y}\right)^{a\alpha};$$

(4 marks)

(b) Find the pdf of Y; (2 marks)

(c) Find the mean and variance of Y; (4 marks)

(d) Find the value at risk of Y; (2 marks)

(e) Find the expected shortfall of Y; (4 marks)

(f) If $y_1, y_2, \ldots, y_n$ is a random sample on Y derive the set of all possible maximum likelihood estimates of a and K. (4 marks)

(Total marks: 20)

END OF EXAMINATION PAPER