MATH68181

Three hours

To be supplied by the Examinations Office: Mathematical Formula Tables and Statistical Tables

THE UNIVERSITY OF MANCHESTER

EXTREME VALUES AND FINANCIAL RISK

Examiner:

Answer QUESTION 1, QUESTION 2 and any FOUR of the remaining questions.

Electronic calculators are permitted provided that cannot store text.

---

1. Let $e_t = \sigma_t Z_t$ denote a financial time series, where $\{\sigma_t\}$ denotes a *volatility process* and $\{Z_t\}$ denotes an *innovation process*. Define each of the following models fully:

(a) e_t follows the ARCH (q) model; (2 marks)

(b) e_t follows the GARCH (p, q) model; (2 marks)

(c) e_t follows the the NGARCH model. (2 marks)

(d) Suppose now that $\{Z_t\}$ are independent standard normal random variables. Then show that

$$E\left(e_t\right) = 0$$

and

$$Var\left(e_t\right) = E\left[\sigma_t^2\right]$$

for each of the models in (a)-(c). (5 marks)

Suppose now $\{e_t\}$ are stationary with $E\left[\sigma_t^2\right] = \sigma^2$ for all t. Find explicit expressions for σ^2 if

(e) e_t follows the ARCH (q) model; (3 marks)

(f) e_t follows the GARCH (p, q) model; (3 marks)

(g) e_t follows the the NGARCH model. (3 marks)

(Total marks: 20)

2. State the conditions in full for $C(u_1, u_2)$, $0 \le u_1, u_2 \le 1$ to be a copula. (4 marks)

Show each of the following is a copula.

(a) the copula defined by $C(u_1, u_2) = \min(u_1, u_2)$. (4 marks)

(b) the copula defined by

$$C(u_1, u_2) = u_1 u_2 \exp\left[-\theta \log u_1 \log u_2\right]$$

for $0 < \theta \le 1$. (4 marks)

(c) the Farlie-Gumbel-Morgenstern copula defined by

$$C(u_1, u_2) = u_1 u_2 \left[1 + \phi (1 - u_1)(1 - u_2)\right]$$

for $-1 \le \phi \le 1$. (4 marks)

(d) the Burr copula defined by

$$C(u_1, u_2) = u_1 + u_2 - 1 + \left[(1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1\right]^{-\alpha}$$

for $\alpha > 0$. (4 marks)

(Total marks: 20)

Turn to next page

3. Consider a bivariate distribution specified by the joint survival function

$$\overline{F}(x, y) = \exp\left[-\left(x^a + y^a\right)^{1/a}\right]$$

for $x > 0$, $y > 0$ and $a \geq 1$.

(a) show that the distribution is a bivariate extreme value distribution; (6 marks)

(b) derive the joint cumulative distribution function; (2 marks)

(c) derive the conditional cumulative distribution function of Y given $X = x$; (4 marks)

(d) derive the conditional cumulative distribution function of X given $Y = y$; (4 marks)

(e) derive the joint probability density function. (4 marks)

(Total marks: 20)

Turn to next page

4. Suppose that the stock returns of a company have an exponential distribution with random rate parameter λ.

(a) determine the actual distribution of the stock returns if λ has a uniform distribution on $[a, b]$, where both a and b are unknown parameters; (5 marks)

(b) find the mean of the distribution in part (a); (5 marks)

(c) find the variance of the distribution in part (a); (5 marks)

(d) if $x_1, x_2, \ldots, x_n$ is a random sample from the actual distribution in part (a), derive the equations for the maximum likelihood estimates of the parameters a and b. (5 marks)

(Total marks: 20)

Turn to next page

5. (a) Suppose $X_1, X_2, \ldots, X_n$ is a random sample with cdf $F(\cdot)$. State the Extremal Types Theorem for $M_n = \max(X_1, X_2, \ldots, X_n)$. You must clearly specify the cdfs of each of the three extreme value distributions. (6 marks)

(b) State in full the necessary and sufficient conditions for $F(\cdot)$ to belong to the domain of attraction of each of the three extreme value distributions. (6 marks)

(c) Consider a class of distributions defined by the cdf

$$F(x) = K \int_0^{G(x)} t^{a-1}(1-t)^{b-1} \exp(-ct) dt,$$

and the pdf

$$f(x) = K g(x) G(x)^{a-1} \left\{1 - G(x)\right\}^{b-1} \exp\left\{-c\, G(x)\right\},$$

where $a > 0$, K is a constant, $G(\cdot)$ is a cdf and $g(x) = dG(x)/dx$. Show that F belongs to the same max domain of attraction as G. (8 marks)

(Total marks: 20)

Turn to next page

6. Determine the domain of attraction (if there is one) for each of the following distributions:

(a) the exponentiated extension distribution given by the cdf

$$F(x) = 1 - \exp\left[1 - (1 + \lambda x)^{\alpha}\right], x > 0, \alpha > 0, \lambda > 0;$$

(4 marks)

(b) the inverse exponentiated exponential distribution given by the cdf

$$F(x) = 1 - \left[1 - \exp\left(-\frac{\lambda}{x}\right)\right]^{\alpha}, x > 0, \alpha > 0, \lambda > 0;$$

(4 marks)

(c) the Poisson distribution given by the pmf

$$p(k) = \frac{\lambda^k \exp(-\lambda)}{k!}, \lambda > 0, k = 0, 1, \ldots;$$

(4 marks)

(d) the Bernoulli distribution given by the pmf

$$p(k) = \begin{cases} 1 - p, & \text{if } k = 0, \\ p, & \text{if } k = 1; \end{cases}$$

(4 marks)

(e) the discrete Weibull distribution given by the cdf

$$F(x) = 1 - q^{(x+1)^a}, 0 < q < 1, a > 1, x = 0, 1, 2, \ldots.$$

(4 marks)

(Total marks: 20)

Turn to next page

7. If X is an absolutely continuous random variable with cdf $F(\cdot)$, then define $\mathrm{VaR}_p(X)$, the Value at Risk of X, and $\mathrm{ES}_p(X)$, the Expected Shortfall of X, explicitly. (3 marks)

If X is a normal random variable with mean μ and standard deviation σ, derive explicit expressions for $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$. (3 marks)

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a normal distribution with unknown mean μ and unknown standard deviation σ.

(a) Write down the joint likelihood function of μ and σ; (1 marks)

(b) Show that the maximum likelihood estimator (mle) of μ is the sample mean $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$; (1 marks)

(c) Show that the mle of σ is $S = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}$; (1 marks)

(d) Deduce the mles of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$; (2 marks)

(e) Show that the mle of $\mathrm{VaR}_p(X)$ is biased; (5 marks)

(f) Show that the mle of $\mathrm{ES}_p(X)$ is also biased. (4 marks)

(Total marks: 20)

Turn to next page

8. Suppose a portfolio is made up of α assets where α is unknown. Suppose also that the price of each asset, say X_i, $i = 1, 2, \ldots, \alpha$, is an exponential random variable with an unknown parameter λ. Then $Y = \min\left(X_1, \ldots, X_\alpha\right)$ will be the price of the least expensive asset. Do the following:

(a) find the cdf of Y; (4 marks)

(b) find the pdf of Y; (2 marks)

(c) find the mean and variance of Y; (4 marks)

(d) find the value at risk of Y; (2 marks)

(e) find the expected shortfall of Y; (4 marks)

(f) if $y_1, y_2, \ldots, y_n$ is a random sample on Y derive the set of all possible maximum likelihood estimates of α and λ. (4 marks)

(Total marks: 20)

END OF EXAMINATION PAPER