

Two hours

To be supplied by the Examinations Office: Mathematical Formula Tables and Statistical Tables

THE UNIVERSITY OF MANCHESTER

EXTREME VALUES AND FINANCIAL RISK

Examiner:

Answer any FOUR questions.

Electronic calculators are permitted provided that cannot store text.

1. Consider a bivariate distribution specified by the joint survival function

$$\bar{F}(x, y) = \exp \left[- (x^a + y^a)^{1/a} \right]$$

for $x > 0$, $y > 0$ and $a \geq 1$.

- (a) show that the distribution is a bivariate extreme value distribution; (6 marks)
- (b) derive the joint cumulative distribution function; (2 marks)
- (c) derive the conditional cumulative distribution function of Y given $X = x$; (4 marks)
- (d) derive the conditional cumulative distribution function of X given $Y = y$; (4 marks)
- (e) derive the joint probability density function. (4 marks)

(Total marks: 20)

2. Suppose that the stock returns of a company have an exponential distribution with random rate parameter λ .

- (a) determine the actual distribution of the stock returns if λ has a uniform distribution on $[a, b]$, where both a and b are unknown parameters; (5 marks)
- (b) find the mean of the distribution in part (a); (5 marks)
- (c) find the variance of the distribution in part (a); (5 marks)
- (d) if x_1, x_2, \dots, x_n is a random sample from the actual distribution in part (a), derive the equations for the maximum likelihood estimates of the parameters a and b . (5 marks)

(Total marks: 20)

3. (a) Suppose X_1, X_2, \dots, X_n is a random sample with cdf $F(\cdot)$. State the Extremal Types Theorem for $M_n = \max(X_1, X_2, \dots, X_n)$. You must clearly specify the cdfs of each of the three extreme value distributions. (6 marks)

(b) State in full the necessary and sufficient conditions for $F(\cdot)$ to belong to the domain of attraction of each of the three extreme value distributions. (6 marks)

(c) Consider a class of distributions defined by the cdf

$$F(x) = K \int_0^{G(x)} t^{a-1} (1-t)^{b-1} \exp(-ct) dt,$$

and the pdf

$$f(x) = K g(x) G(x)^{a-1} \{1 - G(x)\}^{b-1} \exp\{-c G(x)\},$$

where $a > 0$, K is a constant, $G(\cdot)$ is a cdf and $g(x) = dG(x)/dx$. Show that F belongs to the same max domain of attraction as G . (8 marks)

(Total marks: 20)

4. Determine the domain of attraction (if there is one) for each of the following distributions:

(a) the exponentiated extension distribution given by the cdf

$$F(x) = 1 - \exp[1 - (1 + \lambda x)^\alpha], x > 0, \alpha > 0, \lambda > 0;$$

(4 marks)

(b) the inverse exponentiated exponential distribution given by the cdf

$$F(x) = 1 - \left[1 - \exp\left(-\frac{\lambda}{x}\right)\right]^\alpha, x > 0, \alpha > 0, \lambda > 0;$$

(4 marks)

(c) the Poisson distribution given by the pmf

$$p(k) = \frac{\lambda^k \exp(-\lambda)}{k!}, \lambda > 0, k = 0, 1, \dots;$$

(4 marks)

(d) the Bernoulli distribution given by the pmf

$$p(k) = \begin{cases} 1 - p, & \text{if } k = 0, \\ p, & \text{if } k = 1; \end{cases}$$

(4 marks)

(e) the discrete Weibull distribution given by the cdf

$$F(x) = 1 - q^{(x+1)^a}, 0 < q < 1, a > 1, x = 0, 1, 2, \dots$$

(4 marks)

(Total marks: 20)

5. If X is an absolutely continuous random variable with cdf $F(\cdot)$, then define $\text{VaR}_p(X)$, the Value at Risk of X , and $\text{ES}_p(X)$, the Expected Shortfall of X , explicitly. (3 marks)

If X is a normal random variable with mean μ and standard deviation σ , derive explicit expressions for $\text{VaR}_p(X)$ and $\text{ES}_p(X)$. (3 marks)

Suppose X_1, X_2, \dots, X_n is a random sample from a normal distribution with unknown mean μ and unknown standard deviation σ .

(a) Write down the joint likelihood function of μ and σ ; (1 marks)

(b) Show that the maximum likelihood estimator (mle) of μ is the sample mean $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$; (1 marks)

(c) Show that the mle of σ is $S = \sqrt{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2}$; (1 marks)

(d) Deduce the mles of $\text{VaR}_p(X)$ and $\text{ES}_p(X)$; (2 marks)

(e) Show that the mle of $\text{VaR}_p(X)$ is biased; (5 marks)

(f) Show that the mle of $\text{ES}_p(X)$ is also biased. (4 marks)

(Total marks: 20)

6. Suppose a portfolio is made up of α assets where α is unknown. Suppose also that the price of each asset, say X_i , $i = 1, 2, \dots, \alpha$, is an exponential random variable with an unknown parameter λ . Then $Y = \min(X_1, \dots, X_\alpha)$ will be the price of the least expensive asset. Do the following:

- (a) find the cdf of Y ; (4 marks)
- (b) find the pdf of Y ; (2 marks)
- (c) find the mean and variance of Y ; (4 marks)
- (d) find the value at risk of Y ; (2 marks)
- (e) find the expected shortfall of Y ; (4 marks)
- (f) if y_1, y_2, \dots, y_n is a random sample on Y derive the set of all possible maximum likelihood estimates of α and λ . (4 marks)

(Total marks: 20)