Three hours

To be supplied by the Examinations Office: Mathematical Formula Tables and Statistical Tables

THE UNIVERSITY OF MANCHESTER

EXTREME VALUES AND FINANCIAL RISK

Examiner:

Answer QUESTION 1, QUESTION 2 and any FOUR of the remaining questions.

University-approved calculators may be used

END OF EXAMINATION PAPER

1. Consider a bivariate distribution specified by the joint survival function

$$\overline{F}(x, y) = \Pr(X > x, Y > y) = \exp\left[-\frac{\theta y^2}{x + y} + \theta y - x - y\right]$$

for $x > 0$ and $y > 0$.

(a) Show that the distribution is a bivariate extreme value distribution. (6 marks)

(b) Derive the joint cdf. (2 marks)

(c) Derive the conditional cdf of Y given $X = x$. (4 marks)

(d) Derive the conditional cdf of X given $Y = y$. (4 marks)

(e) Derive the joint pdf. (4 marks)

(Total marks: 20)

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2. Suppose that stock returns of a company can be modeled by an exponential distribution with random rate parameter λ.

(a) Determine the actual distribution of stock returns if λ has an exponential distribution with rate parameter a, where a is an unknown parameter. (5 marks)

(b) Find the mean of the distribution in part (a). (5 marks)

(c) Find the variance of the distribution in part (a). (5 marks)

(b) If $x_1, x_2, \ldots, x_n$ is a random sample from the actual distribution in part (a), derive the equation for the maximum likelihood estimate of the parameter a. (5 marks)

(Total marks: 20)

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3. Suppose $X_1, X_2, \ldots, X_n$ is a random sample with cdf $F(\cdot)$. State the Extremal Types Theorem for $M_n = \max(X_1, X_2, \ldots, X_n)$. You must clearly specify the cdfs of each of the three extreme value distributions. (6 marks)

State in full the necessary and sufficient conditions for $F(\cdot)$ to belong to the domain of attraction of each of the three extreme value distributions. (6 marks)

Consider a class of distributions defined by the cdf

$$F(x) = \frac{1}{\Gamma(a)} \int_{-\infty}^{x} \{-\log[1 - G(y)]\}^{a-1} g(y) dy$$

and the pdf

$$f(x) = \frac{1}{\Gamma(a)} \{-\log[1 - G(x)]\}^{a-1} g(x)$$

where $a > 0$, $G(\cdot)$ is a valid cdf, and $g(x) = dG(x)/dx$. Show that F belongs to the same max domain of attraction as G. (8 marks)

(Total marks: 20)

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4. Suppose a portfolio is made up of three assets with X, Y and Z denoting the corresponding prices. Suppose also that the joint distribution of X, Y and Z is specified by the survival function

$$\overline{F}\left(x, y, z\right) = \left[1 + \frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right]^{-d}$$

for $x > 0$, $y > 0$, $z > 0$, $a > 0$, $b > 0$, $c > 0$, and $d > 0$. Find the following:

(a) The cdf of $M = \max(X, Y, Z)$. (4 marks)

(b) The pdf of M. (2 marks)

(c) The nth moment of M. (4 marks)

(d) The cdf of $L = \min(X, Y, Z)$. (4 marks)

(e) The pdf of L. (2 marks)

(f) The nth moment of L. (4 marks)

(Total marks: 20)

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5. State the domain of attraction (if there is one) for each of the following distributions:

(i) The exponentiated exponential distribution given by the cdf

$$F(x) = \left[1 - \exp(-x)\right]^{\alpha},$$

for $x > 0$ and $\alpha > 0$. (4 marks)

(ii) The exponentiated exponential geometric distribution given by the cdf

$$F(x) = \frac{\{1 - \exp(-\beta x)\}^{\alpha}}{1 - p + p\{1 - \exp(-\beta x)\}^{\alpha}},$$

for $x > 0$, $\alpha > 0$, $\beta > 0$ and $0 < p < 1$. (4 marks)

(iii) The exponential-negative binomial distribution given by the cdf

$$F(x) = 1 - \frac{(1-p)^k \exp(-k\beta x)}{\left[1 - p\exp(-\beta x)\right]^k},$$

for $x > 0$, $k > 0$, $0 < p < 1$ and $\beta > 0$. (4 marks)

(iv) The degenerate distribution given by the pmf

$$p(x) = \begin{cases} 1, & \text{if } x = x_0, \\ 0, & \text{if } x \neq x_0. \end{cases}$$

(4 marks)

(v) The Poisson distribution given by the pmf

$$p(x) = \frac{\lambda^x \exp(-\lambda)}{x!},$$

for $\lambda > 0$ and $x = 0, 1, \ldots$. (4 marks)

(Total marks: 20)

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6. If X is an absolutely continuous random variable with cdf $F(\cdot)$, then define $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$ explicitly. (3 marks)

If X is a uniform $[a, b]$ random variable derive explicit expressions for $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$. (3 marks)

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from uniform $[a, b]$, where both a and b are unknown.

(i) Write down the joint likelihood function of a and b. (1 marks)

(ii) Show that the maximum likelihood estimator (mle) of a is $\widehat{a} = \min(X_1, X_2, \ldots, X_n)$. (1 marks)

(iii) Show that the mle of b is $\widehat{b} = \max(X_1, X_2, \ldots, X_n)$. (1 marks)

(iv) Deduce the mles of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$. (2 marks)

(v) Show that the mle of $\mathrm{VaR}_p(X)$ is biased. (5 marks)

(vi) Show that the mle of $\mathrm{ES}_p(X)$ is also biased. (4 marks)

(Total marks: 20)

7. Suppose a portfolio is made up of α assets where α is unknown. Suppose also that the price of each asset, say X_i, $i = 1, 2, \ldots, \alpha$, is an exponential random variable with an unknown parameter λ. Then $Y = \max(X_1, \ldots, X_\alpha)$ will be the price of the most expensive asset.

(i) Find the cdf of Y. (4 marks)

(ii) Find the pdf of Y. (2 marks)

(iii) Find the nth moment of Y. (4 marks)

(iv) Find the value at risk of Y. (2 marks)

(v) Find the expected shortfall of Y. (4 marks)

(vi) If $y_1, y_2, \ldots, y_n$ is a random sample on Y derive the equations for the maximum likelihood estimates of α and λ. (4 marks)

(Total marks: 20)

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8. Let X_i denote the stock price of a product at day i. Suppose X_i are independent exponential random variables with parameters λ_i. Find explicit expressions for the following:

(a) $\Pr\left(X_1 < X_2\right)$. (4 marks)

(b) $\Pr\left(X_1 < X_2 < X_3\right)$. (4 marks)

(c) $\Pr\left(X_1 < X_2 < X_3 < X_4\right)$. (4 marks)

You must show full working for each probability.

Now deduce the expression for the general probability $\Pr\left(X_1 < X_2 < \cdots < X_k\right)$. (4 marks)

Calculate the values of $\Pr(X_1 < X_2)$, $\Pr(X_1 < X_2 < X_3)$, $\Pr(X_1 < X_2 < X_3 < X_4)$, and $\Pr(X_1 < X_2 < \cdots < X_k)$ for the particular case $\lambda_i = \lambda$ for all i. (4 marks)

(Total marks: 20)

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