The Extremal types theorem

Lemma 1. *If G is max-stable, then there exist real-valued functions $a(s) > 0$ and $b(s)$, defined for $s > 0$, such that*

$$G^n(a(s)x + b(s)) = G(x).$$

Proof. Since G is max-stable, there exist $a_n > 0$ and b_n such that

$$G^s(a_n x + b_n) = G(x) \xrightarrow{d} G(x).$$

Thus $G^{\lfloor ns \rfloor}(a_{\lfloor ns \rfloor} x + b_{\lfloor ns \rfloor}) = G(x)$, and we deduce that

$$G^n(a_{\lfloor ns \rfloor} x + b_{\lfloor ns \rfloor}) = \exp\Big\{\frac{n}{\lfloor ns \rfloor} \lfloor ns \rfloor \log G(a_{\lfloor ns \rfloor} x + b_{\lfloor ns \rfloor})\Big\} \xrightarrow{d} G^{1/s}(x).$$

Since $G^{1/s}$ is non-degenerate, the lemma from lectures gives that there exist $a(s) > 0$ and $b(s)$ such that $G(a(s)x + b(s)) = G^{1/s}(x)$, so $G^s(a(s)x + b(s)) = G(x)$. □

Theorem 2 (Extremal types theorem). *Let (X_n) be independent with distribution function F and let $X_{(n)} = \max_{1 \le i \le n} X_{(i)}$. If there exist constants $a_n > 0$ and b_n and a non-degenerate distribution function G such that*

$$\mathbb{P}\Big(\frac{X_{(n)} - b_n}{a_n} \le x\Big) \xrightarrow{d} G(x),$$

then G must be of the same type as one of the three extreme value *classes below:*

Type I *(Fréchet):* $G_{1,\alpha}(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \exp(-x^{-\alpha}) & \text{if } x > 0 \end{cases}$ *for some* $\alpha > 0$

Type II *(Negative Weibull):* $G_{2,\alpha}(x) = \begin{cases} \exp\{-(-x)^{\alpha}\} & \text{if } x < 0 \\ 1 & \text{if } x \ge 0 \end{cases}$ *for some* $\alpha > 0$

Type III *(Gumbel):* $G_3(x) = \exp(-e^{-x})$ *for* $x \in \mathbb{R}$.

Conversely, any distribution function of the same type as one of these extreme value classes can appear as such a limit.

Proof. It suffices to show that the class of max-stable distribution functions coincides with the set of distribution functions of the same type as the three given extreme value classes. To check that the given distribution functions are max-stable, it suffices to observe that if $a_n = n^{1/\alpha}$, $b_n = 0$, then

$$G^n_{1,\alpha}(a_n x + b_n) = \begin{cases} 0 & \text{if } x \leq 0 \\ \exp\{-n(a_n x + b_n)^{-\alpha}\} & \text{if } x > 0 \end{cases} = G_{1,\alpha}(x).$$

Similarly, if $a_n = n^{-1/\alpha}$, $b_n = 0$, then

$$G^n_{2,\alpha}(a_n x + b_n) = \begin{cases} \exp\{-n(-a_n x - b_n)^{\alpha}\} & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} = G_{2,\alpha}(x).$$

Finally, if $a_n = 1$, $b_n = \log n$, then

$$G_3(a_n x + b_n) = \exp\{-ne^{-(a_n x + b_n)}\} = \exp(-e^{-x}).$$

Conversely, suppose G is max-stable, so by Lemma 1 we can write $G^s(a(s)x + b(s)) = G(x)$. It follows that for $0 < G(x) < 1$,

$$-\log\{-\log G(a(s)x + b(s))\} - \log s = \log\{-\log G(x)\}.$$

The max-stability property with $n = 2$ gives that $G^2(ax + b) = G(x)$ for some $a > 0$ and $b \in \mathbb{R}$, which means G cannot have a jump at $x_- = \sup\{x : G(x) = 0\}$ or $x_+ = \inf\{x : G(x) = 1\}$ if these are finite. Thus the non-decreasing function $\psi(x) = -\log\{-\log G(x)\}$ is such that

$$\lim_{x \to x_-} \psi(x) = -\infty, \qquad \lim_{x \to x_+} \psi(x) = \infty.$$

Therefore ψ has an inverse function $U(y) = \inf\{x \in \mathbb{R} : \psi(x) \geq y\}$, defined for all $y \in \mathbb{R}$, and since $\psi(a(s)x + b(s)) - \log s = \psi(x)$, it follows that

$$\begin{aligned} U(y) &= \inf\{x : \psi(a(s)x + b(s)) - \log s \geq y\} \\ &= \frac{1}{a(s)}\{\inf\{x' : \psi(x') \geq y + \log s\} - b(s)\} \\ &= \frac{U(y + \log s) - b(s)}{a(s)}. \end{aligned}$$

Subtracting this equation for $y = 0$,

$$\frac{U(y + \log s) - U(\log s)}{a(s)} = U(y) - U(0),$$

and writing $z = \log s$, $\tilde{a}(z) = a(e^z)$ and $\tilde{U}(y) = U(y) - U(0)$,

$$\tilde{U}(y + z) - \tilde{U}(z) = \tilde{U}(y)\tilde{a}(z) \tag{1}$$

for all $y, z \in \mathbb{R}$. Interchanging y and z and subtracting,

$$\tilde{U}(y)\{1-\tilde{a}(z)\} = \tilde{U}(z)\{1-\tilde{a}(y)\}. \tag{2}$$

Two cases are possible:

i) $\tilde{a}(z_0) \neq 1$ for some $z_0 > 0$. Then $\tilde{a}(z) \neq 1$ for all $z > 0$, because otherwise there exists $z > 0$ such that $\tilde{U}(z) = 0$. But this would mean that $\tilde{U}(y+z) = \tilde{U}(y)$ for all y, by (1), so $U(y+z) = U(y)$ for all $y \in \mathbb{R}$, a contradiction. Fixing $z > 0$, writing $c = \tilde{U}(z)/\{1-\tilde{a}(z)\}$ and noting from (2) that this is constant, we have from (1) that

$$c(1-\tilde{a}(y+z)) - c(1-\tilde{a}(z)) = c(1-\tilde{a}(y))\tilde{a}(z),$$

so that

$$\tilde{a}(y+z) = \tilde{a}(y)\tilde{a}(z)$$

for all $y \in \mathbb{R}$. But $\tilde{a}$ is monotone, since $\tilde{U}(y) = c\{1-\tilde{a}(y)\}$ from (2), and the only non-zero solutions that are monotone and not identically equal to 1 are $\tilde{a}(y) = e^{\rho y}$ for some $\rho \neq 0$ (check). But then

$$\psi^{-1}(y) = U(y) = \nu + c(1-e^{\rho y})$$

where $\nu = U(0)$. Since ψ^{-1} is non-decreasing, we must have $c < 0$ if $\rho > 0$ and $c > 0$ if $\rho < 0$, so in fact ψ^{-1} is continuous and strictly increasing. Hence

$$x = \psi^{-1}(\psi(x)) = \nu + c(1-e^{\rho\psi(x)}) = \nu + c[1-\{-\log G(x)\}^{-\rho}],$$

so

$$G(x) = \exp\Big\{-\Big(1-\frac{x-\nu}{c}\Big)^{-1/\rho}\Big\}$$

for $0 < G(x) < 1$. From the continuity of G at any finite endpoints, we see that G is of Type I, with $\alpha = 1/\rho$, if $\rho > 0$, and of Type II, with $\alpha = -1/\rho$, if $\rho < 0$.

ii) $\tilde{a}(z) = 1$ for all $z > 0$. But then, from (1),

$$\tilde{U}(y+z) = \tilde{U}(y) + \tilde{U}(z),$$

for which the only non-constant non-decreasing solutions are $\tilde{U}(y) = \rho y$ for some $\rho > 0$. Thus

$$\psi^{-1}(y) = U(y) = \nu + \rho y,$$

where $\nu = U(0)$, and since this is continuous and strictly increasing,

$$x = \psi^{-1}(\psi(x)) = \rho\psi(x) + \nu = -\rho\log\{-\log G(x)\} + \nu.$$

Hence $G(x) = \exp\{-e^{-(x-\nu)/\rho}\}$ for $0 < G(x) < 1$, and since G has no jump at any finite endpoint, G is of Type III. $\square$