MATH4/68181: Extreme values and financial risk
Semester 1
Solutions to problem sheet 9

1. Let X denote the actual stock return. The pdf of X is

$$\begin{aligned} f_X(x) &= \int_0^\infty \lambda \exp(-\lambda x) a \exp(-a\lambda) d\lambda \\ &= a \int_0^\infty \lambda \exp\left\{-\lambda(x+a)\right\} d\lambda \\ &= \frac{a}{(x+a)^2}. \end{aligned}$$

If $x_1, x_2, \ldots, x_n$ is a random sample on X then the likelihood function is

$$L(a) = a^n \prod_{i=1}^n \left(x_i + a\right)^{-2}.$$

The log-likelihood function is

$$\log L = n \log a - 2 \sum_{i=1}^n \log\left(x_i + a\right).$$

The derivative with respect to a is

$$\frac{d \log L}{da} = \frac{n}{a} - 2 \sum_{i=1}^n \frac{1}{x_i + a}.$$

So, the mle of a is the root of the equation

$$\frac{n}{a} = 2 \sum_{i=1}^n \frac{1}{x_i + a}.$$

2. Let X denote the actual stock return. The pdf of X is

$$\begin{aligned} f_X(x) &= \frac{1}{b-a} \int_a^b \lambda \exp(-\lambda x) d\lambda \\ &= \frac{1}{b-a} \left\{ \left[\lambda \frac{\exp(-\lambda x)}{-x} \right]_a^b + \frac{1}{x} \int_a^b \exp(-\lambda x) d\lambda \right\} \\ &= \frac{1}{b-a} \left\{ -\frac{b \exp(-bx) - a \exp(-ax)}{x} - \frac{\exp(-bx) - \exp(-ax)}{x^2} \right\} \\ &= \frac{(xa+1)\exp(-ax) - (xb+1)\exp(-bx)}{x^2(b-a)}. \end{aligned}$$

If $x_1, x_2, \ldots, x_n$ is a random sample on X then the likelihood function is

$$L(a,b) = (b-a)^{-n} \prod_{i=1}^n \frac{(x_i a + 1)\exp\left(-a x_i\right) - (x_i b + 1)\exp\left(-b x_i\right)}{x_i^2}.$$

The log-likelihood function is

$$\log L(a,b) = -n\log(b-a) + \sum_{i=1}^{n} \log\left[(x_i a+1)\exp(-ax_i) - (x_i b+1)\exp(-bx_i)\right] - 2\sum_{i=1}^{n}\log x_i.$$

The partial derivatives with respect to a and b are

$$\frac{\partial \log L}{\partial a} = \frac{n}{b-a} - a\sum_{i=1}^{n} \frac{x_i^2 \exp(-ax_i)}{(x_i a+1)\exp(-ax_i) - (x_i b+1)\exp(-bx_i)}$$

and

$$\frac{\partial \log L}{\partial b} = -\frac{n}{b-a} - b\sum_{i=1}^{n} \frac{x_i^2 \exp(-bx_i)}{(x_i a+1)\exp(-ax_i) - (x_i b+1)\exp(-bx_i)}.$$

So, the mles of a and b are the simultaneous solutions of the equations

$$\frac{n}{b-a} = a\sum_{i=1}^{n} \frac{x_i^2 \exp(-ax_i)}{(x_i a+1)\exp(-ax_i) - (x_i b+1)\exp(-bx_i)}$$

and

$$-\frac{n}{b-a} = b\sum_{i=1}^{n} \frac{x_i^2 \exp(-bx_i)}{(x_i a+1)\exp(-ax_i) - (x_i b+1)\exp(-bx_i)}.$$

3. Let X denote the actual stock return. The pdf of X is

$$\begin{aligned} f_X(x) &= a\int_0^1 \lambda^a \exp(-\lambda x)d\lambda \\ &= ax^{-a-1}\gamma(a+1,x), \end{aligned}$$

where

$$\gamma(a,x) = \int_0^x t^{a-1}\exp(-t)dt$$

is the incomplete gamma function. If $x_1, x_2, \ldots, x_n$ is a random sample on X then the likelihood function is

$$L(a) = a^n \left(\prod_{i=1}^{n} x_i\right)^{-a-1} \prod_{i=1}^{n} \gamma(a+1,x_i).$$

The log-likelihood function is

$$\log L(a) = n\log a - (a+1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log\gamma(a+1,x_i).$$

The derivative with respect to a is

$$\frac{d\log L}{da} = \frac{n}{a} - \sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\frac{1}{\gamma(a+1,x_i)}\frac{\partial\gamma(a+1,x_i)}{\partial a}.$$

So, the mle of a is

$$-\frac{n}{a} + \sum_{i=1}^{n}\log x_i = \sum_{i=1}^{n}\frac{1}{\gamma(a+1,x_i)}\frac{\partial\gamma(a+1,x_i)}{\partial a}.$$

4. Let X denote the actual stock return. The pdf of X is

$$\begin{aligned} f_X(x) &= aK^a \int_K^\infty \lambda^{-a} \exp(-\lambda x) d\lambda \\ &= aK^a x^{a-1} \Gamma(1-a, Kx), \end{aligned}$$

where

$$\Gamma(a,x) = \int_x^\infty t^{a-1} \exp(-t) dt$$

is the complementary incomplete gamma function. If $x_1, x_2, \ldots, x_n$ is a random sample on X then the likelihood function is

$$L(a,K) = a^n K^{an} \left(\prod_{i=1}^n x_i \right)^{a-1} \prod_{i=1}^n \Gamma\left(1-a, Kx_i\right).$$

The log-likelihood function is

$$\log L(a,K) = n \log a + an \log K + (a-1) \sum_{i=1}^n \log x_i + \sum_{i=1}^n \log \Gamma\left(1-a, Kx_i\right).$$

The partial derivatives with respect to a and K are

$$\frac{\partial \log L}{\partial a} = \frac{n}{a} + n \log K + \sum_{i=1}^n \log x_i + \sum_{i=1}^n \frac{1}{\left(1-a, Kx_i\right)} \frac{\partial \left(1-a, Kx_i\right)}{\partial a}$$

and

$$\frac{\partial \log L}{\partial K} = \frac{na}{K} + \sum_{i=1}^n \frac{1}{\left(1-a, Kx_i\right)} \frac{\partial \left(1-a, Kx_i\right)}{\partial K}.$$

So, the mles of a and K are the simultaneous solutions of the equations

$$\frac{n}{a} + n \log K + \sum_{i=1}^n \log x_i = -\sum_{i=1}^n \frac{1}{\left(1-a, Kx_i\right)} \frac{\partial \left(1-a, Kx_i\right)}{\partial a}$$

and

$$\frac{na}{K} = -\sum_{i=1}^n \frac{1}{\left(1-a, Kx_i\right)} \frac{\partial \left(1-a, Kx_i\right)}{\partial K}.$$