# MATH4/68181: Extreme values and financial risk
# Semester 1
# Solutions to problem sheet for Week 5

1. The cdf of $M = \max(X, Y)$ is

$$\begin{aligned} F_M(m) &= \Pr(X < m, Y < m) \\ &= 1 - \Pr(X > m) - \Pr(Y > m) + \Pr(X > m, Y > m) \\ &= 1 - \left[1 + \frac{m}{a}\right]^{-c} - \left[1 + \frac{m}{b}\right]^{-c} + \left[1 + \frac{m}{a} + \frac{m}{b}\right]^{-c}. \end{aligned}$$

2. Differentiating

$$F_M(m) = 1 - \left[1 + \frac{m}{a}\right]^{-c} - \left[1 + \frac{m}{b}\right]^{-c} + \left[1 + \frac{m}{a} + \frac{m}{b}\right]^{-c}.$$

with respect to $m$ gives the pdf of $M$ as

$$f_M(m) = \frac{c}{a}\left[1 + \frac{m}{a}\right]^{-c-1} + \frac{c}{b}\left[1 + \frac{m}{b}\right]^{-c-1} - c\left(\frac{1}{a} + \frac{1}{b}\right)\left[1 + \frac{m}{a} + \frac{m}{b}\right]^{-c-1}.$$

3. The $n$th moment of $M$ can be calculated as

$$\begin{aligned} E\left(M^n\right) &= \frac{c}{a}\int_0^\infty m^n \left[1 + \frac{m}{a}\right]^{-c-1} dm + \frac{c}{b}\int_0^\infty m^n \left[1 + \frac{m}{b}\right]^{-c-1} dm \\ &\quad -c\int_0^\infty m^n \left(\frac{1}{a} + \frac{1}{b}\right)\left[1 + \frac{m}{a} + \frac{m}{b}\right]^{-c-1} dm \\ &= \frac{c}{a}a^{n+1}\int_0^1 x^{c-n-1}(1-x)^n dx + \frac{c}{b}b^{n+1}\int_0^1 x^{c-n-1}(1-x)^n dx \\ &\quad -c\left(\frac{1}{a} + \frac{1}{b}\right)\left(\frac{1}{a} + \frac{1}{b}\right)^{-n-1}\int_0^1 x^{c-n-1}(1-x)^n dx \\ &= ca^n B(c-n, n+1) + cb^n B(c-n, n+1) - c\left(\frac{1}{a} + \frac{1}{b}\right)^{-n} B(c-n, n+1). \end{aligned}$$

4. The mean of $M$ is

$$\begin{aligned} E\left(M\right) &= caB(c-1, 2) + cbB(c-1, 2) - c\left(\frac{1}{a} + \frac{1}{b}\right)^{-1} B(c-1, 2) \\ &= \frac{c\left(a^2 + b^2 + ab\right)}{a+b} B(c-1, 2). \end{aligned}$$

5. The variance of $M$ is

$$\begin{aligned} Var\left(M\right) &= ca^2 B(c-2, 3) + cb^2 B(c-2, 3) - c\left(\frac{1}{a} + \frac{1}{b}\right)^{-2} B(c-2, 3) - E^2(M) \\ &= \frac{c\left[\left(a^2 + b^2\right)(a+b)^2 - a^2b^2\right)}{(a+b)^2} B(c-2, 3) - \frac{c^2\left(a^2 + b^2 + ab\right)^2}{(a+b)^2} B^2(c-1, 2). \end{aligned}$$

6. The cdf of $L = \min(X, Y)$ is

$$\begin{aligned} F_L(l) &= 1 - \Pr(L > l) \\ &= 1 - \Pr(X > l, Y > l) \\ &= 1 - \left[1 + \frac{l}{a} + \frac{l}{b}\right]^{-c}. \end{aligned}$$

7. Differentiating

$$F_L(l) = 1 - \left[1 + \frac{l}{a} + \frac{l}{b}\right]^{-c}.$$

with respect to $l$ gives the pdf of $L$ as

$$f_L(l) = c\left(\frac{1}{a} + \frac{1}{b}\right)\left[1 + \frac{l}{a} + \frac{l}{b}\right]^{-c-1}.$$

8. The $n$th moment of $L$ can be calculated as

$$\begin{aligned} E\left(L^n\right) &= c\left(\frac{1}{a} + \frac{1}{b}\right)\int_0^\infty l^n \left[1 + \frac{l}{a} + \frac{l}{b}\right]^{-c-1} dl \\ &= c\left(\frac{1}{a} + \frac{1}{b}\right)\left(\frac{1}{a} + \frac{1}{b}\right)^{-n-1}\int_0^1 x^{c-n-1}(1-x)^n dx \\ &= c\left(\frac{1}{a} + \frac{1}{b}\right)^{-n} B(c-n, n+1). \end{aligned}$$

9. The mean of $L$ is

$$E\left(L\right) = \frac{abc}{a+b} B(c-1, 2).$$

10. The variance of $L$ is

$$Var\left(L\right) = \frac{a^2 b^2 c}{(a+b)^2} B(c-2, 3) - \frac{a^2 b^2 c^2}{(a+b)^2} B^2(c-1, 2).$$