MATH4/68181: Extreme values and financial risk
Semester 1
Solutions to problem sheet for Week 4

1. The cdf of X is

$$\begin{aligned} F_X(x) &= \Pr(X \leq x) \\ &= \Pr\left(\max\left(X_1, \ldots, X_\alpha\right) \leq x\right) \\ &= \Pr\left(X_1 \leq x, \ldots, X_\alpha \leq x\right) \\ &= \Pr\left(X_1 \leq x\right) \cdots \Pr\left(X_\alpha \leq x\right) \\ &= [1 - \exp(-\lambda x)] \cdots [1 - \exp(-\lambda x)] \\ &= [1 - \exp(-\lambda x)]^\alpha . \end{aligned}$$

2. Differentiating

$$F_X(x) = [1 - \exp(-\lambda x)]^\alpha$$

with respect to x gives the pdf of X as

$$f_X(x) = \alpha\lambda \exp(-\lambda x) [1 - \exp(-\lambda x)]^{\alpha-1} .$$

3. The nth moment of X can be calculated as

$$\begin{aligned} E\left(X^n\right) &= \alpha\lambda \int_0^\infty x^n \exp(-\lambda x) [1 - \exp(-\lambda x)]^{\alpha-1} dx \\ &= (-1)^n \alpha \lambda^{-n} \int_0^1 (\log y)^n [1 - y]^{\alpha-1} dy \\ &= (-1)^n \alpha \lambda^{-n} \left. \frac{\partial^n}{\partial \beta^n} \int_0^1 y^\beta [1 - y]^{\alpha-1} dy \right|_{\beta=0} \\ &= (-1)^n \alpha \lambda^{-n} \left. \frac{\partial^n}{\partial \beta^n} B(\beta + 1, \alpha) \right|_{\beta=0} . \end{aligned}$$

4. The mean of X can be calculated as

$$\begin{aligned} E(X) &= -\alpha\lambda^{-1} \left. \frac{\partial}{\partial \beta} B(\beta + 1, \alpha) \right|_{\beta=0} \\ &= -\alpha\lambda^{-1} \left. \left\{ \frac{\Gamma(\alpha)\Gamma^{'}(\beta+1)}{\Gamma(\alpha+\beta+1)} - \frac{\Gamma(\alpha)\Gamma(\beta+1)\Gamma^{'}(\alpha+\beta+1)}{\Gamma^2(\alpha+\beta+1)} \right\} \right|_{\beta=0} \\ &= -\alpha\lambda^{-1} \left\{ \frac{\Gamma^{'}(1)}{\alpha} - \frac{\Gamma(\alpha)\Gamma^{'}(\alpha+1)}{\Gamma^2(\alpha+1)} \right\} . \end{aligned}$$

5. Since

$$E\left(X^2\right) = -\alpha\lambda^{-2} \left. \frac{\partial^2}{\partial \beta^2} B(\beta + 1, \alpha) \right|_{\beta=0}$$

$$
\begin{aligned}
&= -\alpha\lambda^{-2}\left\{ \frac{\Gamma(\alpha)\Gamma''(\beta+1)}{\Gamma(\alpha+\beta+1)} - \frac{2\Gamma(\alpha)\Gamma'(\beta+1)\Gamma'(\alpha+\beta+1)}{\Gamma^2(\alpha+\beta+1)} \right. \\
&\quad \left. - \frac{\Gamma(\alpha)\Gamma(\beta+1)\Gamma''(\alpha+\beta+1)}{\Gamma^2(\alpha+\beta+1)} + \frac{2\Gamma(\alpha)\Gamma(\beta+1)\left(\Gamma'(\alpha+\beta+1)\right)^2}{\Gamma^3(\alpha+\beta+1)} \right\}\Bigg|_{\beta=0} \\
&= -\alpha\lambda^{-2}\left\{ \frac{\Gamma''(1)}{\alpha} - \frac{2\Gamma(\alpha)\Gamma'(1)\Gamma'(\alpha+1)}{\Gamma^2(\alpha+1)} \right. \\
&\quad \left. - \frac{\Gamma(\alpha)\Gamma''(\alpha+1)}{\Gamma^2(\alpha+1)} + \frac{2\Gamma(\alpha)\left(\Gamma'(\alpha+1)\right)^2}{\Gamma^3(\alpha+1)} \right\},
\end{aligned}
$$

the variance of X is

$$
\begin{aligned}
Var\,(X) &= -\alpha\lambda^{-2}\left\{ \frac{\Gamma''(1)}{\alpha} - \frac{2\Gamma(\alpha)\Gamma'(1)\Gamma'(\alpha+1)}{\Gamma^2(\alpha+1)} \right. \\
&\quad \left. - \frac{\Gamma(\alpha)\Gamma''(\alpha+1)}{\Gamma^2(\alpha+1)} + \frac{2\Gamma(\alpha)\left(\Gamma'(\alpha+1)\right)^2}{\Gamma^3(\alpha+1)} \right\} - E^2(X).
\end{aligned}
$$

6. Setting

$$
[1 - \exp(-\lambda x)]^\alpha = p
$$

gives

$$
\mathrm{VaR}_p(X) = -\frac{1}{\lambda}\log\left(1 - p^{1/\alpha}\right).
$$

7. The expected shortfall is

$$
\begin{aligned}
\mathrm{ES}_p(X) &= -\frac{1}{\lambda p}\int_0^p \log\left(1 - v^{1/\alpha}\right) dv \\
&= \frac{1}{\lambda p}\int_0^p \sum_{i=0}^{\infty} \frac{v^{i/\alpha}}{i} dv \\
&= \frac{1}{\lambda p}\sum_{i=0}^{\infty} \int_0^p \frac{v^{i/\alpha}}{i} dv \\
&= \frac{1}{\lambda p}\sum_{i=0}^{\infty} \frac{p^{i/\alpha+1}}{i\,(i/\alpha+1)}.
\end{aligned}
$$

8. The likelihood function is

$$
L(\alpha,\lambda) = \alpha^n\lambda^n \exp\left(-\lambda\sum_{i=1}^n x_i\right)\left\{\prod_{i=1}^n \left[1 - \exp\left(-\lambda x_i\right)\right]\right\}^{\alpha-1}.
$$

The log-likelihood function is

$$
\log L = n\log(\alpha\lambda) - \lambda\sum_{i=1}^n x_i + (\alpha-1)\sum_{i=1}^n \log\left[1 - \exp\left(-\lambda x_i\right)\right].
$$

The partial derivatives with respect to α and λ are

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log \left[1 - \exp\left(-\lambda x_i\right) \right]$$

and

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + (\alpha - 1) \sum_{i=1}^{n} \frac{x_i \exp\left(-\lambda x_i\right)}{1 - \exp\left(-\lambda x_i\right)}.$$

Setting these to zero, we obtain the mle of α as

$$\widehat{\alpha} = -\frac{n}{\displaystyle\sum_{i=1}^{n} \log \left[1 - \exp\left(-\lambda x_i\right) \right]}.$$

The mle of λ is the root of the equation

$$\frac{n}{\lambda} - \sum_{i=1}^{n} x_i - \left(\frac{n}{\displaystyle\sum_{i=1}^{n} \log \left[1 - \exp\left(-\lambda x_i\right) \right]} + 1 \right) \sum_{i=1}^{n} \frac{x_i \exp\left(-\lambda x_i\right)}{1 - \exp\left(-\lambda x_i\right)}.$$