MATH4/68181: Extreme values and financial risk
Semester 1
Solutions to problem sheet for Week 6 and Week 7

1. Setting

$$1 - \exp(-\lambda x) = p$$

gives

$$\mathrm{VaR}_p(X) = -\frac{1}{\lambda}\log(1-p).$$

So,

$$\begin{aligned}
\mathrm{ES}_p(X) &= -\frac{1}{p\lambda}\int_0^p \log(1-v)dv \\
&= -\frac{1}{p\lambda}\left\{[\log(1-v)v]_0^p + \int_0^p \frac{v}{1-v}dv\right\} \\
&= -\frac{1}{p\lambda}\left\{\log(1-p)p - p + \int_0^p \frac{1}{1-v}dv\right\} \\
&= -\frac{1}{p\lambda}\left\{\log(1-p)p - p - \log(1-p)\right\}.
\end{aligned}$$

2. Setting

$$x^a = p$$

gives

$$\mathrm{VaR}_p(X) = p^{1/a}.$$

So,

$$\begin{aligned}
\mathrm{ES}_p(X) &= \frac{1}{p}\int_0^p v^{1/a}dv \\
&= \frac{1}{p(1/a+1)}p^{1/a+1} \\
&= \frac{p^{1/a}}{1/a+1}.
\end{aligned}$$

3. Setting

$$\frac{x-a}{b-a} = p$$

gives

$$\mathrm{VaR}_p(X) = a + p(b-a).$$

So,

$$\begin{aligned}
\mathrm{ES}_p(X) &= \frac{1}{p}\int_0^p [a + v(b-a)]\, dv \\
&= \frac{1}{p}\left[ap + \frac{p^2}{2}(b-a)\right] \\
&= a + \frac{p}{2}(b-a).
\end{aligned}$$

4. Setting

$$1 - \left(\frac{K}{x}\right)^a = p$$

gives

$$\mathrm{VaR}_p(X) = K(1-p)^{-1/a}.$$

So,

$$\begin{aligned}
\mathrm{ES}_p(X) &= \frac{K}{p}\int_0^p (1-v)^{-1/a} dv \\
&= \frac{K}{1-a}\left[(1-p)^{1-1/a} - 1\right].
\end{aligned}$$

5. Setting

$$\Phi(x) = p$$

gives

$$\mathrm{VaR}_p(X) = \Phi^{-1}(p).$$

So,

$$\mathrm{ES}_p(X) = \frac{1}{p}\int_0^p \Phi^{-1}(v) dv.$$

6. Setting

$$\frac{1}{1 + (x/a)^{-b}} = p,$$

we have

$$\begin{aligned}
& 1 + (x/a)^{-b} = \frac{1}{p} \\
\Leftrightarrow\ & (x/a)^{-b} = \frac{1}{p} - 1 = \frac{1-p}{p} \\
\Leftrightarrow\ & \frac{x}{a} = \left(\frac{1-p}{p}\right)^{-1/b} \\
\Leftrightarrow\ & x = a\left(\frac{1-p}{p}\right)^{-1/b},
\end{aligned}$$

giving

$$\mathrm{VaR}_p(X) = ap^{1/b}(1-p)^{-1/b}.$$

So,

$$\begin{aligned}
\mathrm{ES}_p(X) &= \frac{a}{p}\int_0^p v^{1/b}(1-v)^{-1/b}dv \\
&= \frac{a}{p}B_p\left(1/b+1, 1-1/b\right),
\end{aligned}$$

where

$$B_x\left(a,b\right) = \int_0^x t^{a-1}(1-t)^{b-1}dt$$

is the incomplete beta function.

7. Setting

$$1-\left(1+\frac{x}{\lambda}\right)^{-\alpha} = p$$

gives

$$\mathrm{VaR}_p(X) = \lambda\left[(1-p)^{-1/\alpha}-1\right].$$

So,

$$\begin{aligned}
\mathrm{ES}_p(X) &= \frac{\lambda}{p}\int_0^p \left[(1-v)^{-1/\alpha}-1\right]dv \\
&= \frac{\lambda}{p}\left[\frac{1-(1-p)^{1-1/\alpha}}{1-1/\alpha}-p\right].
\end{aligned}$$

8. Setting

$$\exp\left\{-\left(\frac{\sigma}{x}\right)^{\alpha}\right\} = p,$$

we have

$$\begin{aligned}
& -\left(\frac{\sigma}{x}\right)^{\alpha} = \log p \\
\Leftrightarrow\ & \left(\frac{\sigma}{x}\right)^{\alpha} = -\log p \\
\Leftrightarrow\ & \frac{\sigma}{x} = (-\log p)^{1/\alpha} \\
\Leftrightarrow\ & \frac{x}{\sigma} = (-\log p)^{-1/\alpha} \\
\Leftrightarrow\ & x = \sigma(-\log p)^{-1/\alpha},
\end{aligned}$$

giving

$$\mathrm{VaR}_p(X) = \sigma\left[-\log p\right]^{-1/\alpha}.$$

So,

$$\begin{aligned}
\mathrm{ES}_p(X) &= \frac{\sigma}{p}\int_0^p [-\log v]^{-1/\alpha}\,dv \\
&= -\frac{\sigma}{p}\int_{\infty}^{-\log p} x^{-1/\alpha}\exp(-x)dx \text{ [set } x=-\log v] \\
&= \frac{\sigma}{p}\int_{-\log p}^{\infty} x^{-1/\alpha}\exp(-x)dx \\
&= \frac{\sigma}{p}\Gamma\left(1-1/\alpha,-\log p\right),
\end{aligned}$$

where

$$\Gamma\left(a,x\right)=\int_x^{\infty} t^{a-1}\exp(-t)dt$$

is the complementary incomplete gamma function.

9. Setting

$$1-\exp\left\{-\left(\frac{x}{\sigma}\right)^{\alpha}\right\}=p,$$

we have

$$\begin{aligned}
& \exp\left\{-\left(\frac{x}{\sigma}\right)^{\alpha}\right\}=1-p \\
\Leftrightarrow\quad & -\left(\frac{x}{\sigma}\right)^{\alpha}=\log(1-p) \\
\Leftrightarrow\quad & \left(\frac{x}{\sigma}\right)^{\alpha}=-\log(1-p) \\
\Leftrightarrow\quad & \frac{x}{\sigma}=[\log(1-p)]^{1/\alpha} \\
\Leftrightarrow\quad & x=\sigma\left[\log(1-p)\right]^{1/\alpha},
\end{aligned}$$

giving

$$\mathrm{VaR}_p(X)=\sigma\left[-\log(1-p)\right]^{1/\alpha}.$$

So,

$$\begin{aligned}
\mathrm{ES}_p(X) &= \frac{\sigma}{p}\int_0^p [-\log(1-v)]^{1/\alpha}\,dv \\
&= \frac{\sigma}{p}\int_0^{-\log(1-p)} x^{1/\alpha}\exp(-x)dx \text{ [set } x=-\log(1-v)] \\
&= \frac{\sigma}{p}\gamma\left(1+1/\alpha,-\log(1-p)\right),
\end{aligned}$$

where

$$\gamma\left(a,x\right)=\int_0^{x} t^{a-1}\exp(-t)dt$$

is the incomplete gamma function.

10. Setting

$$1-\left(1-\frac{cx}{k}\right)^{1/c}=p$$

gives

$$\mathrm{VaR}_p(X)=\frac{k}{c}\left[1-(1-p)^c\right].$$

So,

$$\begin{aligned}\mathrm{ES}_p(X) &= \frac{k}{pc}\int_0^p\left[1-(1-v)^c\right]dv \\ &= \frac{k}{pc}\left\{p+\frac{(1-p)^{c+1}-1}{c+1}\right\}.\end{aligned}$$

11. For this distribution,

$$\mathrm{VaR}_p(X)=\frac{p^\lambda-(1-p)^\lambda}{\lambda}$$

and

$$\mathrm{ES}_p(X)=\frac{1}{p\lambda}\int_0^p\left(v^\lambda-(1-v)^\lambda\right)dv=\frac{1}{p\lambda(\lambda+1)}\left(p^{\lambda+1}+(1-p)^{\lambda+1}-1\right).$$

12. For this distribution,

$$\mathrm{VaR}_p(X)=\frac{p^\beta-(1-p)^\gamma}{\delta}$$

and

$$\mathrm{ES}_p(X)=\frac{1}{p\delta}\int_0^p\left(v^\beta-(1-v)^\gamma\right)dv=\frac{1}{p\delta}\left(\frac{p^{\beta+1}}{\beta+1}+\frac{(1-p)^{\gamma+1}-1}{\gamma+1}\right).$$

13. For this distribution,

$$\mathrm{VaR}_p(X)=\frac{Cp^\alpha}{(1-p)^\beta}$$

and

$$\mathrm{ES}_p(X)=\frac{C}{p}\int_0^p\frac{v^\alpha}{(1-v)^\beta}dv=\frac{C}{p}B_p(\alpha+1,1-\beta).$$