MATH4/68181: Extreme values and financial risk
Semester 1
Solutions to problem sheet 5

1. If $X_i \sim Exp(\lambda_i)$, $i = 1, 2$ are independent random variables then

$$\begin{aligned}
\Pr\left(X_1 < X_2\right) &= \int_0^\infty \left[1 - \exp\left(-\lambda_1 x_2\right)\right] \lambda_2 \exp\left(-\lambda_2 x_2\right) dx_2 \\
&= 1 - \int_0^\infty \lambda_2 \exp\left(-\lambda_1 x_2 - \lambda_2 x_2\right) dx_2 \\
&= 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}.
\end{aligned}$$

2. If $X_i \sim Exp(\lambda_i)$, $i = 1, 2, 3$ are independent random variables then

$$\begin{aligned}
\Pr\left(X_1 < X_2 < X_3\right) &= \int_0^\infty \int_{x_1}^\infty \int_{x_2}^\infty \lambda_1 \lambda_2 \lambda_3 \exp\left(-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_3\right) dx_3 dx_2 dx_1 \\
&= \int_0^\infty \int_{x_1}^\infty \lambda_1 \lambda_2 \exp\left(-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_2\right) dx_2 dx_1 \\
&= \frac{\lambda_1 \lambda_2}{\lambda_2 + \lambda_3} \int_0^\infty \exp\left(-\lambda_1 x_1 - \lambda_2 x_1 - \lambda_3 x_1\right) dx_1 \\
&= \frac{\lambda_1 \lambda_2}{\left(\lambda_2 + \lambda_3\right)\left(\lambda_1 + \lambda_2 + \lambda_3\right)}.
\end{aligned}$$

3. If $X_i \sim Exp(\lambda_i)$, $i = 1, 2, 3, 4$ are independent random variables then

$$\begin{aligned}
& \Pr\left(X_1 < X_2 < X_3 < X_4\right) \\
= & \int_0^\infty \int_{x_1}^\infty \int_{x_2}^\infty \int_{x_3}^\infty \lambda_1 \lambda_2 \lambda_3 \lambda_4 \exp\left(-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_3 - \lambda_4 x_4\right) dx_4 dx_3 dx_2 dx_1 \\
= & \int_0^\infty \int_{x_1}^\infty \int_{x_2}^\infty \lambda_1 \lambda_2 \lambda_3 \exp\left(-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_3 - \lambda_4 x_3\right) dx_3 dx_2 dx_1 \\
= & \frac{\lambda_1 \lambda_2 \lambda_3}{\left(\lambda_3 + \lambda_4\right)} \int_0^\infty \int_{x_1}^\infty \exp\left(-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_2 - \lambda_4 x_2\right) dx_2 dx_1 \\
= & \frac{\lambda_1 \lambda_2 \lambda_3}{\left(\lambda_3 + \lambda_4\right)\left(\lambda_2 + \lambda_3 + \lambda_4\right)} \int_0^\infty \exp\left(-\lambda_1 x_1 - \lambda_2 x_1 - \lambda_3 x_1 - \lambda_4 x_1\right) dx_2 dx_1 \\
= & \frac{\lambda_1 \lambda_2 \lambda_3}{\left(\lambda_3 + \lambda_4\right)\left(\lambda_2 + \lambda_3 + \lambda_4\right)\left(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4\right)}.
\end{aligned}$$

4. If $X_i \sim Exp(\lambda_i)$, $i = 1, 2, \ldots, k$ are independent random variables then the general formula will be

$$\Pr\left(X_1 < X_2 < \cdots < X_k\right) = \frac{\lambda_1 \lambda_2 \cdots \lambda_k}{\left(\lambda_k + \lambda_{k-1}\right)\left(\lambda_k + \lambda_{k-1} + \lambda_{k-2}\right) \cdots \left(\lambda_k + \lambda_{k-1} + \cdots + \lambda_1\right)}.$$

This can be proved by induction. Please try that yourself.

5. If $X_i \sim N(\mu_i, \sigma_i^2)$, $i = 1, 2$ are independent random variables then

$$\begin{aligned}
\Pr\left(X_1 < X_2\right) &= \Pr\left(X_1 - X_2 < 0\right) \\
&= \Pr\left(N\left(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2\right) < 0\right) \\
&= \Pr\left(N\left(0,1\right) < \frac{\mu_2 - \mu_1}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right) \\
&= \Phi\left(\frac{\mu_2 - \mu_1}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right).
\end{aligned}$$

6. If X_i, $i = 1, 2$ are independent Pareto random variables as stated in the question then

$$\begin{aligned}
\Pr\left(X_1 < X_2\right) &= \int_{\max(K,L)}^{\infty} \left[1 - \left(\frac{K}{x_2}\right)^a\right] \frac{bL^b}{x_2^{b+1}} dx_2 \\
&= \int_{\max(K,L)}^{\infty} \frac{bL^b}{x_2^{b+1}} dx_2 - \int_{\max(K,L)}^{\infty} \frac{bK^a L^b}{x_2^{a+b+1}} dx_2 \\
&= \left[\frac{L}{\max(K,L)}\right]^b - \frac{bK^a L^b}{(a+b)(\max(K,L))^{a+b}}.
\end{aligned}$$

7. If X_i, $i = 1, 2$ are independent Rayleigh random variables as stated in the question then

$$\begin{aligned}
\Pr\left(X_1 < X_2\right) &= \int_0^{\infty} \left[1 - \exp\left(-\frac{x_2^2}{2\sigma_1^2}\right)\right] \frac{x_2}{\sigma_2^2} \exp\left(-\frac{x_2^2}{2\sigma_2^2}\right) dx_2 \\
&= \int_0^{\infty} \frac{x_2}{\sigma_2^2} \exp\left(-\frac{x_2^2}{2\sigma_2^2}\right) dx_2 - \int_0^{\infty} \frac{x_2}{\sigma_2^2} \exp\left\{-\frac{x_2^2}{2}\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)\right\} dx_2 \\
&= 1 - \frac{1}{\sigma_2^2}\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{-1} \\
&= \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}.
\end{aligned}$$

8. If $X_i \sim uniform[a_i, b_i]$, $i = 1, 2$ are independent uniform random variables then

$$\begin{aligned}
\Pr\left(X_1 < X_2\right) &= \int_{\max(a_1,a_2)}^{\min(b_1,b_2)} \frac{x_2 - a_1}{b_1 - a_1} \frac{1}{b_2 - a_2} dx_2 + \int_{\min(b_1,b_2)}^{b_2} \frac{1}{b_2 - a_2} dx_2 \\
&= \frac{\left[\min\left(b_1, b_2\right) - a_1\right]^2 - \left[\max\left(a_1, a_2\right) - a_1\right]^2}{2\left(b_1 - a_1\right)\left(b_2 - a_2\right)} + \frac{b_2 - \min\left(b_1, b_2\right)}{b_2 - a_2}.
\end{aligned}$$