MATH4/68181: Extreme values and financial risk
Semester 1
Solutions to problem sheet for Week 3

1. For the Bernoulli (p) distribution,

$$p(k) = \begin{cases} 1-p, & \text{if } k = 0, \\ p, & \text{if } k = 1. \end{cases}$$

So,

$$\frac{\Pr(X = k)}{1 - F(k-1)} = \begin{cases} 1-p, & \text{if } k = 0, \\ 1, & \text{if } k = 1. \end{cases}$$

Hence, there can be no sequences $a_n > 0$ and b_n such that $(M_n - b_n)/a_n$ has a non-degenerate limiting distribution.

2. For the degenerate distribution,

$$p(k) = \begin{cases} 1, & \text{if } k = k_0, \\ 0, & \text{if } k \neq k_0. \end{cases}$$

So,

$$F(k) = \begin{cases} 1, & \text{if } k \geq k_0, \\ 0, & \text{if } k < k_0, \end{cases}$$

and

$$\frac{\Pr(X = k)}{1 - F(k-1)} = \begin{cases} 1/1, & \text{if } k = k_0, \\ 0/1, & \text{if } k_0 < k < k_0 + 1, \\ 0/0, & \text{if } k \geq k_0 + 1, \\ 0/1, & \text{if } k < k_0. \end{cases}$$

Hence, there can be no sequences $a_n > 0$ and b_n such that $(M_n - b_n)/a_n$ has a non-degenerate limiting distribution.

3. For the Yule distribution,

$$p(k) = \rho B(k, \rho + 1), \ k \geq 1.$$

By Stirling's formula,

$$\begin{aligned} p(k) &= \frac{\rho \Gamma(\rho+1)\Gamma(k)}{\Gamma(k+\rho+1)} \\ &\sim \frac{\rho\Gamma(\rho+1)\sqrt{2\pi(k-1)}((k-1)/e)^{k-1}}{\sqrt{2\pi(k+\rho)}((k+\rho)/e)^{k+\rho}} \text{ using Stirling's formula} \\ &\sim \rho\Gamma(\rho+1)\sqrt{\frac{k-1}{k+\rho}} e^{\rho+1} \frac{(k-1)^{k-1}}{(k+\rho)^{k+\rho}} \end{aligned}$$

$$\begin{aligned}
&\sim \rho\Gamma(\rho+1)e^{\rho+1}\frac{1}{(k-1)(k+\rho)^{\rho}}\frac{(k-1)^k}{(k+\rho)^k}\\
&\sim \rho\Gamma(\rho+1)e^{\rho+1}\frac{1}{k^{\rho+1}}\frac{(k-1)^k}{(k+\rho)^k}\\
&\sim \rho\Gamma(\rho+1)e^{\rho+1}\frac{1}{k^{\rho+1}}\frac{\left(1-k^{-1}\right)^k}{\left(1+\rho k^{-1}\right)^k}\\
&\sim \rho\Gamma(\rho+1)e^{\rho+1}\frac{1}{k^{\rho+1}}\frac{e^{-1}}{e^{\rho}}\\
&\sim \rho\Gamma(\rho+1)\frac{1}{k^{\rho+1}}
\end{aligned}$$

as $k \to \infty$. So,

$$\begin{aligned}
1-F(k-1) &= \sum_{i=k}^{\infty} p(i)\\
&= \sum_{i=k}^{\infty} \rho\Gamma(\rho+1)\frac{1}{i^{\rho+1}}\\
&= \rho\Gamma(\rho+1)\sum_{i=k}^{\infty}\frac{1}{i^{\rho+1}}\\
&\sim \rho\Gamma(\rho+1)\int_k^{\infty}\frac{1}{x^{\rho+1}}dx\\
&= \rho\Gamma(\rho+1)\left[\frac{x^{-\rho}}{-\rho}\right]_k^{\infty}\\
&= \frac{\Gamma(\rho+1)}{k^{\rho}}
\end{aligned}$$

as $k \to \infty$, and

$$\frac{\Pr(X=k)}{1-F(k-1)} \sim \frac{\rho}{k}$$

as $k \to \infty$. Hence, there are sequences $a_n > 0$ and b_n such that $(M_n - b_n)/a_n$ has a non-degenerate limiting distribution.

4. For the zeta distribution,

$$p(k) = k^{-s}/\zeta(s),\ k \geq 1,$$

So,

$$\begin{aligned}
1-F(k-1) &= \sum_{i=k}^{\infty} p(i)\\
&= \sum_{i=k}^{\infty}\frac{i^{-s}}{\zeta(s)}\\
&= \frac{1}{\zeta(s)}\sum_{i=k}^{\infty} i^{-s}
\end{aligned}$$

$$\begin{aligned}
&\sim \frac{1}{\zeta(s)} \int_k^\infty x^{-s} ds \\
&= \frac{1}{\zeta(s)} \left[\frac{x^{1-s}}{1-s} \right]_k^\infty \\
&= \frac{k^{1-s}}{(s-1)\zeta(s)}
\end{aligned}$$

as $k \to \infty$, and

$$\frac{\Pr(X=k)}{1-F(k-1)} \sim \frac{s-1}{k}$$

as $k \to \infty$. Hence, there are sequences $a_n > 0$ and b_n such that $(M_n - b_n)/a_n$ has a non-degenerate limiting distribution.

5. For the Gauss-Kuzmin distribution,

$$p(k) = -\log_2 \left[1 - (k+1)^{-2} \right], \ k \geq 1,$$

and

$$F(k) = 1 - \log_2 \left[\frac{k+2}{k+1} \right], \ k \geq 1.$$

Using the fact $\partial \log_a z / \partial z = 1/(z \log a)$,

$$\begin{aligned}
\frac{\Pr(X=k)}{1-F(k-1)} &= \frac{-\log_2 \left[1-(k+1)^{-2} \right]}{\log_2 \left[\dfrac{k+1}{k} \right]} \\
&= \frac{-\log_2 \left[\dfrac{k^2+2k}{(k+1)^2} \right]}{\log_2 \left[\dfrac{k+1}{k} \right]} \\
&= \frac{-\log_2 \left(k^2+2k \right) + \log_2 (k+1)^2}{\log_2 (k+1) - \log_2 k} \\
&\sim \frac{-\dfrac{2k+2}{(\log 2)\left(k^2+2k\right)} + \dfrac{2(k+1)}{(\log 2)(k+1)^2}}{\dfrac{1}{(\log 2)(k+1)} - \dfrac{1}{(\log 2) k}} \\
&= \frac{-\dfrac{2(k+1)}{k(k+2)} + \dfrac{2}{k+1}}{-\dfrac{1}{(k+1)k}} \\
&= 2 \frac{(k+1)^2 - k(k+2)}{k+2} \\
&= \frac{2}{k+2}
\end{aligned}$$

as $k \to \infty$. Hence, there are sequences $a_n > 0$ and b_n such that $(M_n - b_n)/a_n$ has a non-degenerate limiting distribution.

6. For the discrete Lindley distribution,

$$p(x) = \frac{p^x}{1+\theta} \left\{ \theta(1-2p) + (1-p)(1+\theta x) \right\},$$

where $p = \exp(-\theta)$, for $\theta > 0$ and $x = 0, 1, 2, \ldots$. The corresponding cdf can be calculated as

$$\begin{aligned}
F(x-1) &= \frac{\theta(1-2p)+1-p}{1+\theta} \sum_{y=0}^{x-1} p^y + \frac{\theta(1-p)}{1+\theta} \sum_{y=0}^{x-1} y p^y \\
&= \frac{\theta(1-2p)+1-p}{1+\theta} \sum_{y=0}^{x-1} p^y + \frac{\theta p(1-p)}{1+\theta} \frac{d}{dp} \sum_{y=0}^{x-1} p^y \\
&= \frac{\theta(1-2p)+1-p}{1+\theta} \left[\frac{1-p^x}{1-p} \right] + \frac{\theta p(1-p)}{1+\theta} \frac{d}{dp} \left[\frac{1-p^x}{1-p} \right] \\
&= \frac{\theta(1-2p)+1-p}{1+\theta} \left[\frac{1-p^x}{1-p} \right] + \frac{\theta p(1-p)}{1+\theta} \frac{-(1-p)xp^{x-1}+1-p^x}{(1-p)^2} \\
&= \frac{\left[\theta(1-2p)+1-p\right](1-p^x)}{(1+\theta)(1-p)} + \frac{\theta p \left[-(1-p)xp^{x-1}+1-p^x \right]}{(1+\theta)(1-p)} \\
&= \frac{(1+\theta)(1-p)+p^x(p-1)(x\theta+\theta+1)}{(1+\theta)(1-p)} \\
&= 1 - \frac{1+\theta+\theta x}{1+\theta} p^x.
\end{aligned}$$

So,

$$\begin{aligned}
\frac{\Pr(X=x)}{1-F(x-1)} &= \frac{\theta(1-2p)+(1-p)(1+\theta x)}{1+\theta+\theta x} \\
&= \frac{\dfrac{\theta(1-2p)+1-p}{x} + (1-p)\theta}{\dfrac{1+\theta}{x} + \theta} \\
&\to 1-p.
\end{aligned}$$

Hence, there can be no sequences $a_n > 0$ and b_n such that $(M_n - b_n)/a_n$ has a non-degenerate limiting distribution.

7. For the discrete Weibull distribution,

$$F(x) = 1 - q^{(x+1)^a},$$

where $0 < q < 1$, for $a > 0$ and $x = 0, 1, 2, \ldots$. The corresponding pmf is

$$p(x) = q^{x^a} - q^{(x+1)^a}.$$

So,

$$\frac{\Pr(X=x)}{1-F(x-1)} = 1 - q^{(x+1)^a - x^a}.$$

Note that

$$
\begin{aligned}
x^a - (x+1)^a &= x^a - x^a\left(1+\frac{1}{x}\right)^a \\
&= x^a\left[1-\left(1+\frac{1}{x}\right)^a\right] \\
&= x^a\left[1-1-a\frac{1}{x}-\frac{a(a-1)}{2!}\frac{1}{x^2}-\cdots\right] \\
&\to \begin{cases} -\infty, & \text{if } a>1, \\ -1, & \text{if } a=1, \\ 0, & \text{if } a<1. \end{cases}
\end{aligned}
$$

Hence,

$$
\frac{\Pr(X=x)}{1-F(x-1)} \to \begin{cases} 1, & \text{if } a>1, \\ 1-q, & \text{if } a=1, \\ 0, & \text{if } a<1. \end{cases}
$$

Hence, there can be sequences $a_n > 0$ and b_n such that $(M_n - b_n)/a_n$ has a non-degenerate limiting distribution if $a < 1$. There can be no sequences $a_n > 0$ and b_n such that $(M_n - b_n)/a_n$ has a non-degenerate limiting distribution if $a \geq 1$.