MATH4/68181: Extreme values and financial risk
Semester 1
Solutions to problem sheet for Week 1

1) The likelihood function is

$$L(\sigma) = \sigma^{-n} \exp\left(-\frac{1}{\sigma}\sum_{i=1}^{n} x_i\right) \exp\left\{-\sum_{i=1}^{n} \exp\left(-\frac{x_i}{\sigma}\right)\right\}.$$

The log-likelihood function is

$$\log L(\sigma) = -n \log \sigma^{-n} - \frac{1}{\sigma}\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \exp\left(-\frac{x_i}{\sigma}\right).$$

The derivative with respect to σ is

$$\frac{d \log L(\sigma)}{d\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i - \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i \exp\left(-\frac{x_i}{\sigma}\right).$$

So, the mle of σ is the root of the equation

$$n\sigma = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i \exp\left(-\frac{x_i}{\sigma}\right).$$

2) The likelihood function is

$$L(\lambda, \sigma) = \lambda^n \sigma^{n\lambda} \left(\prod_{i=1}^{n} x_i\right)^{-\lambda-1} \exp\left(-\sigma^{\lambda}\sum_{i=1}^{n} x_i^{-\lambda}\right).$$

The log-likelihood function is

$$\log L(\lambda, \sigma) = n \log \lambda + n\lambda \log \sigma - (\lambda + 1)\sum_{i=1}^{n} \log x_i - \sigma^{\lambda}\sum_{i=1}^{n} x_i^{-\lambda}.$$

The partial derivatives with respect to λ and σ are

$$\frac{d \log L}{d\lambda} = \frac{n}{\lambda} + n \log \sigma - \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^{\lambda} \log\left(\frac{\sigma}{x_i}\right)$$

and

$$\frac{d \log L}{d\sigma} = \frac{n\lambda}{\sigma} - \lambda\sigma^{\lambda-1}\sum_{i=1}^{n} x_i^{-\lambda}.$$

Setting these to zero and rearranging, we obtain the mle of σ as

$$\widehat{\sigma} = \left[\frac{n}{\displaystyle\sum_{i=1}^{n} x_i^{-\lambda}}\right]^{1/\lambda}.$$

The mle of λ is the root of the equation

$$\frac{n}{\lambda} - \sum_{i=1}^{n} \log x_i = -\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}} \sum_{i=1}^{n} x_i^{-\lambda} \log x_i.$$

3) The likelihood function is

$$L(\lambda, \sigma) = \lambda^n \sigma^{-n\lambda} \left(\prod_{i=1}^{n} x_i \right)^{\lambda - 1} \exp\left(-\sigma^{-\lambda} \sum_{i=1}^{n} x_i^{\lambda} \right).$$

The log-likelihood function is

$$\log L(\lambda, \sigma) = n \log \lambda - n\lambda \log \sigma + (\lambda - 1) \sum_{i=1}^{n} \log x_i - \sigma^{-\lambda} \sum_{i=1}^{n} x_i^{\lambda}.$$

The partial derivatives with respect to λ and σ are

$$\frac{d \log L}{d\lambda} = \frac{n}{\lambda} - n \log \sigma + \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{x_i}{\sigma} \right)^{\lambda} \log\left(\frac{x_i}{\sigma} \right)$$

and

$$\frac{d \log L}{d\sigma} = -\frac{n\lambda}{\sigma} + \lambda \sigma^{-\lambda - 1} \sum_{i=1}^{n} x_i^{\lambda}.$$

Setting these to zero and rearranging, we obtain the mle of σ as

$$\widehat{\sigma} = \left[\frac{n}{\sum_{i=1}^{n} x_i^{\lambda}} \right]^{-1/\lambda}.$$

The mle of λ is the root of the equation

$$\frac{n}{\lambda} + \sum_{i=1}^{n} \log x_i = \frac{n}{\sum_{i=1}^{n} x_i^{\lambda}} \sum_{i=1}^{n} x_i^{\lambda} \log x_i.$$

4) The likelihood function is

$$L(\lambda) = \left(\prod_{i=1}^{n} (1 - \lambda x_i) \right)^{1/\lambda - 1}.$$

The log-likelihood function is

$$\log L(\lambda) = (1/\lambda - 1) \sum_{i=1}^{n} \log (1 - \lambda x_i).$$

The derivative with respect to λ is

$$\frac{d \log L(\lambda)}{d\lambda} = -\lambda^{-2} \sum_{i=1}^{n} \log\left(1 - \lambda x_i\right) - (1/\lambda - 1) \sum_{i=1}^{n} \frac{x_i}{1 - \lambda x_i}.$$

So, the mle of λ is the root of the equation

$$\left(\lambda^2 - \lambda\right) \sum_{i=1}^{n} \frac{x_i}{1 - \lambda x_i} = \sum_{i=1}^{n} \log\left(1 - \lambda x_i\right).$$