MATH4/68181: Extreme values and financial risk
Semester 1
Solutions to problem sheet 2

There are many financial indices that take the form of ratios. Some of the most commonly known examples are:

1. Current ratio defined by Current assets (X)/Current liabilities (Y).
2. Sales margin defined by (Sales (X) - Costs (Y))/Sales (X).
3. Changes in capital employed defined by (Closing capital (Y) - Opening capital (X))/Opening capital (X).
4. Interest cover defined by (Earnings (X) + Interests paid (Y))/Earnings (X).
5. Liabilities ratio defined by Liabilities (X)/(Equity (Y) + Liabilities (X)).
6. Financial leverage ratio defined by Liabilities (X)/(Total capital (Y) - Liabilities (X)).

Suppose X and Y are independent Pareto random variables with cdfs specified by

$$F_X(x) = 1 - (K/x)^a, \ x \geq K$$

and

$$F_Y(y) = 1 - (L/y)^b, \ y \geq L,$$

respectively, where $K > 0$, $L > 0$, $a > 0$ and $b > 0$. The corresponding pdfs are

$$f_X(x) = aK^a/x^{a+1}, \ x \geq K$$

and

$$f_Y(y) = bL^b/y^{b+1}, \ y \geq L,$$

respectively, where $K > 0$, $L > 0$, $a > 0$ and $b > 0$. We now derive the cdf and pdf of each of the mentioned ratios.

1) Let Z = Current ratio = Current assets (X)/Current liabilities (Y). We first determine the cdf of Z. We consider two cases: $z < K/L$ and $z \geq K/L$. If $z < K/L$ then

$$F_Z(z) = \int_K^\infty \int_{x/z}^\infty \frac{abK^aL^b}{x^{a+1}y^{b+1}} dydx = \int_K^\infty \frac{aK^aL^b}{x^{a+1}(x/z)^b} dx = \frac{aL^bz^b}{(a+b)K^b}.$$

If $z \geq K/L$ then

$$F_Z(z) = 1 - \int_L^\infty \int_{zy}^\infty \frac{abK^aL^b}{x^{a+1}y^{b+1}} dxdy = 1 - \int_L^\infty \frac{bK^aL^b}{y^{b+1}(zy)^a} dx = 1 - \frac{bK^a}{(a+b)L^az^a}.$$

So, the cdf of Z is

$$F_Z(z) = \begin{cases} \dfrac{aL^b z^b}{(a+b)K^b}, & \text{if } z < K/L, \\ 1 - \dfrac{bK^a}{(a+b)L^a z^a}, & \text{if } z \geq K/L. \end{cases}$$

Differentiating with respect to z, we obtain the pdf of Z as

$$f_Z(z) = \begin{cases} \dfrac{abL^b z^{b-1}}{(a+b)K^b}, & \text{if } z < K/L, \\ \dfrac{abK^a}{(a+b)L^a z^{a+1}}, & \text{if } z \geq K/L. \end{cases}$$

2) Let $W = Y/X$. Since $F_W(w) = \Pr(W \leq w) = \Pr(Y/X \leq w) = \Pr(X/Y \geq 1/w) = 1 - F_Z(1/w)$, where $Z = X/Y$, the cdf of W is

$$F_W(w) = \begin{cases} 1 - \dfrac{aL^b}{(a+b)K^b w^b}, & \text{if } w \geq L/K, \\ \dfrac{bK^a w^a}{(a+b)L^a}, & \text{if } w < L/K. \end{cases}$$

Differentiating with respect to w, we obtain the pdf of W as

$$f_W(w) = \begin{cases} \dfrac{abL^b}{(a+b)K^b w^{b+1}}, & \text{if } w \geq L/K, \\ \dfrac{abK^a w^{a-1}}{(a+b)L^a}, & \text{if } z < L/K. \end{cases}$$

Now redefine Z = Sales margin = (Sales (X) - Costs (Y))/Sales (X). Note that $Z = 1 - W$. Since $F_Z(z) = \Pr(Z \leq z) = \Pr(1 - W \leq z) = \Pr(W \geq 1 - z) = 1 - F_W(1 - z)$, the cdf of Z is

$$F_Z(z) = \begin{cases} \dfrac{aL^b}{(a+b)K^b (1-z)^b}, & \text{if } z \leq 1 - L/K, \\ 1 - \dfrac{bK^a (1-z)^a}{(a+b)L^a}, & \text{if } z > 1 - L/K. \end{cases}$$

Differentiating with respect to z, we obtain the pdf of Z as

$$f_Z(z) = \begin{cases} \dfrac{abL^b}{(a+b)K^b (1-z)^{b+1}}, & \text{if } z \leq 1 - L/K, \\ \dfrac{abK^a (1-z)^{a-1}}{(a+b)L^a}, & \text{if } z > 1 - L/K. \end{cases}$$

3) Let Z = changes in capital employed = (Closing capital (Y) - Opening capital (X))/Opening capital (X). With $W = Y/X$ as in 2), we can write $Z = W - 1$. Since $F_Z(z) = \Pr(Z \leq z) = \Pr(W - 1 \leq z) = \Pr(W \geq 1 + z) = F_W(1 + z)$, the cdf of Z is

$$F_Z(z) = \begin{cases} 1 - \dfrac{aL^b}{(a+b)K^b (1+z)^b}, & \text{if } z \geq L/K - 1, \\ \dfrac{bK^a (1+z)^a}{(a+b)L^a}, & \text{if } z < L/K - 1. \end{cases}$$

Differentiating with respect to z, we obtain the pdf of Z as

$$f_Z(z) = \begin{cases} \dfrac{abL^b}{(a+b)K^b(1+z)^{b+1}}, & \text{if } z \geq L/K - 1, \\ \dfrac{abK^a(1+z)^{a-1}}{(a+b)L^a}, & \text{if } z < L/K - 1. \end{cases}$$

4) Let Z = Interest cover = (Earnings (X) + Interests paid (Y))/Earnings (X). With $W = Y/X$ as in 2), we can write $Z = W + 1$. Since $F_Z(z) = \Pr(Z \leq z) = \Pr(W + 1 \leq z) = \Pr(W \geq z - 1) = F_W(z-1)$, the cdf of Z is

$$F_Z(z) = \begin{cases} 1 - \dfrac{aL^b}{(a+b)K^b(z-1)^b}, & \text{if } z \geq L/K + 1, \\ \dfrac{bK^a(z-1)^a}{(a+b)L^a}, & \text{if } z < L/K + 1. \end{cases}$$

Differentiating with respect to z, we obtain the pdf of Z as

$$f_Z(z) = \begin{cases} \dfrac{abL^b}{(a+b)K^b(z-1)^{b+1}}, & \text{if } z \geq L/K + 1, \\ \dfrac{abK^a(z-1)^{a-1}}{(a+b)L^a}, & \text{if } z < L/K + 1. \end{cases}$$

5) Let Z = Liabilities ratio = (X)/(Equity (Y) + Liabilities (X)). With $W = Y/X$ as in 2), we can write $Z = 1/(W+1)$. Since $F_Z(z) = \Pr(1/(W+1) \leq z) = \Pr(W + 1 \geq 1/z) = \Pr(W \geq 1/z - 1) = 1 - F_W(1/z - 1)$, the cdf of Z is

$$F_Z(z) = \begin{cases} \dfrac{aL^b}{(a+b)K^b\left(z^{-1}-1\right)^b}, & \text{if } z \geq K/(L+K), \\ 1 - \dfrac{bK^a\left(z^{-1}-1\right)^a}{(a+b)L^a}, & \text{if } z < K/(L+K). \end{cases}$$

Differentiating with respect to z, we obtain the pdf of Z as

$$f_Z(z) = \begin{cases} \dfrac{abL^b}{z^2(a+b)K^b\left(z^{-1}-1\right)^{b+1}}, & \text{if } z \geq K/(L+K), \\ \dfrac{abK^a\left(z^{-1}-1\right)^{a-1}}{z^2(a+b)L^a}, & \text{if } z < K/(L+K). \end{cases}$$

6) Let Z = Financial leverage ratio = Liabilities (X)/(Total capital (Y) - Liabilities (X)). With $W = Y/X$ as in 2), we can write $Z = 1/(W-1)$. Since $F_Z(z) = \Pr(1/(W-1) \leq z) = \Pr(W - 1 \geq 1/z)\Pr(W > 1) + \Pr(W - 1 \leq 1/z)\Pr(W < 1) = [1 - F_W(1/z+1)][1 - F_W(1)] + F_W(1/z+1)F_W(1) = 1 - F_W(1) + [2F_W(1) - 1]\, F_W(1/z+1)$, the cdf of Z is

$$F_Z(z) = \begin{cases} 1 - F_W(1) + [2F_W(1) - 1]\left[1 - \dfrac{aL^b}{(a+b)K^b\left(z^{-1}+1\right)^b}\right], & \text{if } z \geq K/(L-K), \\ 1 - F_W(1) + [2F_W(1) - 1]\,\dfrac{bK^a\left(z^{-1}+1\right)^a}{(a+b)L^a}, & \text{if } z < K/(L-K), \end{cases}$$

where

$$F_W(1) = \begin{cases} 1 - \dfrac{aL^b}{(a+b)K^b}, & \text{if } 1 \geq L/K, \\ \dfrac{bK^a}{(a+b)L^a}, & \text{if } 1 < L/K. \end{cases}$$

Differentiating with respect to z, we obtain the pdf of Z as

$$f_Z(z) = \begin{cases} \left[1 - 2F_W(1)\right] \dfrac{abL^b}{z^2(a+b)K^b \left(z^{-1}+1\right)^{b+1}}, & \text{if } z \geq K/(L-K), \\ \left[1 - 2F_W(1)\right] \dfrac{abK^a \left(z^{-1}+1\right)^{a-1}}{z^2(a+b)L^a}, & \text{if } z < K/(L-K), \end{cases}$$

where

$$F_W(1) = \begin{cases} 1 - \dfrac{aL^b}{(a+b)K^b}, & \text{if } 1 \geq L/K, \\ \dfrac{bK^a}{(a+b)L^a}, & \text{if } 1 < L/K. \end{cases}$$