MATH4/68181: Extreme values and financial risk
Semester 1
Solutions to problem sheet for Week 10

Solutions to question 1:

(a) We can write

$$\overline{F}(x,y) = \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right] = \exp\left\{-(x+y)\left[\theta\frac{y^2}{(x+y)^2} - \theta\frac{y}{x+y} + 1\right]\right\}.$$

This is in the form of

$$\overline{F}(x,y) = \exp\left[-(x+y)A\left(\frac{y}{x+y}\right)\right]$$

with $A(t) = \theta t^2 - \theta t + 1$.

We now check the conditions for $A(\cdot)$. Clearly, $A(0) = 1$ and $A(1) = 1$.

Also $A(t) \geq 0$ since

$$\begin{aligned} & \theta t^2 - \theta t + 1 \geq 0 \\ \Leftrightarrow\ & \theta\left(t^2 - t\right) + 1 \geq 0 \\ \Leftrightarrow\ & \theta(t-1/2)^2 + 1 - \theta/4 \geq 0, \end{aligned}$$

which always holds.

Also $A(t) \leq 1$ since

$$\begin{aligned} & \theta t^2 - \theta t + 1 \leq 1 \\ \Leftrightarrow\ & \theta t^2 - \theta t \leq 0 \\ \Leftrightarrow\ & \theta t(t-1) \leq 0, \end{aligned}$$

which always holds.

Also $A(t) \geq t$ since

$$\begin{aligned} & \theta t^2 - \theta t + 1 \geq t \\ \Leftrightarrow\ & \theta t^2 - (\theta+1)t + 1 \geq 0 \\ \Leftrightarrow\ & (1-\theta t)(1-t) \geq 0, \end{aligned}$$

which always holds.

Also $A(t) \geq 1 - t$ since

$$\begin{aligned} & \theta t^2 - \theta t + 1 \geq 1 - t \\ \Leftrightarrow\ & \theta t^2 + (1-\theta)t \geq 0 \\ \Leftrightarrow\ & (\theta t - \theta + 1)t \geq 0, \end{aligned}$$

which always holds.

$A^{'}(t) = 2\theta t - \theta$ and $A^{''}(t) = 2\theta > 0$, so $A(\cdot)$ is convex.

(b) the joint cdf is

$$F(x,y) = 1 - \exp(-x) - \exp(-y) + \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right].$$

(c) the derivative of joint cdf with respect to x is

$$\frac{\partial F(x,y)}{\partial x} = \exp(-x) + \left[\frac{\theta y^2}{(x+y)^2} - 1\right] \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right],$$

so the conditional cdf if Y given $X = x$ is

$$F(y|x) = 1 + \left[\frac{\theta y^2}{(x+y)^2} - 1\right] \exp\left[-\frac{\theta y^2}{x+y} + \theta y - y\right].$$

(d) the derivative of joint cdf with respect to y is

$$\frac{\partial F(x,y)}{\partial y} = \exp(-y) + \left[\frac{\theta y^2}{(x+y)^2} - \frac{2\theta y}{x+y} + \theta - 1\right] \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right],$$

so the conditional cdf if X given $Y = y$ is

$$F(x|y) = 1 + \left[\frac{\theta y^2}{(x+y)^2} - \frac{2\theta y}{x+y} + \theta - 1\right] \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x\right].$$

(e) the derivative of joint cdf with respect to x and y is

$$\begin{aligned} f(x,y) &= \frac{\partial F(x,y)}{\partial x \partial y} \\ &= \left\{\left[\frac{\theta y^2}{(x+y)^2} - 1\right]\left[\frac{\theta y^2}{(x+y)^2} - \frac{2\theta y}{x+y} + \theta - 1\right] + \left[\frac{2\theta y}{(x+y)^2} - \frac{2\theta y^2}{(x+y)^3}\right]\right\} \\ &\quad \times \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right]. \end{aligned}$$

(f) the conditional pdf of Y given $X = x$ is

$$\begin{aligned} f(y|x) &= \left\{\left[\frac{\theta y^2}{(x+y)^2} - 1\right]\left[\frac{\theta y^2}{(x+y)^2} - \frac{2\theta y}{x+y} + \theta - 1\right] + \left[\frac{2\theta y}{(x+y)^2} - \frac{2\theta y^2}{(x+y)^3}\right]\right\} \\ &\quad \times \exp\left[-\frac{\theta y^2}{x+y} + \theta y - y\right]. \end{aligned}$$

(g) the conditional pdf of X given $Y = y$ is

$$\begin{aligned} f(x|y) &= \left\{\left[\frac{\theta y^2}{(x+y)^2} - 1\right]\left[\frac{\theta y^2}{(x+y)^2} - \frac{2\theta y}{x+y} + \theta - 1\right] + \left[\frac{2\theta y}{(x+y)^2} - \frac{2\theta y^2}{(x+y)^3}\right]\right\} \\ &\quad \times \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x\right]. \end{aligned}$$

Solutions to question 2:

(a) We can write

$$\overline{F}(x,y) = \exp\left[\frac{\alpha xy}{x+y} - x - y\right] = \exp\left\{-(x+y)\left[1 - \alpha\frac{y}{x+y}\left(1 - \frac{y}{x+y}\right)\right]\right\}.$$

This is in the form of

$$\overline{F}(x,y) = \exp\left[-(x+y)A\left(\frac{y}{x+y}\right)\right]$$

with $A(t) = 1 - \alpha t(1-t)$.

We now check the conditions for $A(\cdot)$. Clearly, $A(0) = 1$ and $A(1) = 1$.

Also $A(t) \geq 0$ since

$$\begin{aligned} & 1 - \alpha t(1-t) \geq 0 \\ \Leftrightarrow\ & \alpha t(1-t) \leq 1, \end{aligned}$$

which always holds.

Also $A(t) \leq 1$ since

$$\begin{aligned} & 1 - \alpha t(1-t) \leq 1 \\ \Leftrightarrow\ & \alpha t(1-t) \geq 0, \end{aligned}$$

which always holds.

Also $A(t) \geq t$ since

$$\begin{aligned} & 1 - \alpha t(1-t) \geq t \\ \Leftrightarrow\ & 1 - t - \alpha t(1-t) \geq 0 \\ \Leftrightarrow\ & (1-t)(1-\alpha t) \geq 0, \end{aligned}$$

which always holds.

Also $A(t) \geq 1 - t$ since

$$\begin{aligned} & 1 - \alpha t(1-t) \geq 1 - t \\ \Leftrightarrow\ & t - \alpha t(1-t) \geq 0 \\ \Leftrightarrow\ & t\left(1 - \alpha(1-t)\right) \geq 0, \end{aligned}$$

which always holds.

$A^{'}(t) = -\alpha + 2\alpha t$ and $A^{''}(t) = 2\alpha > 0$, so $A(\cdot)$ is convex.

(b) the joint cdf is

$$F(x,y) = 1 - \exp(-x) - \exp(-y) + \exp\left[\frac{\alpha xy}{x+y} - x - y\right].$$

(c) the derivative of joint cdf with respect to x is

$$\frac{\partial F(x,y)}{\partial x} = \exp(-x) + \left[-1 + \frac{\alpha y}{x+y} - \frac{\alpha xy}{(x+y)^2}\right] \exp\left[\frac{\alpha xy}{x+y} - x - y\right],$$

so the conditional cdf if Y given $X = x$ is

$$F(y|x) = 1 + \left[-1 + \frac{\alpha y}{x+y} - \frac{\alpha xy}{(x+y)^2}\right] \exp\left[\frac{\alpha xy}{x+y} - y\right].$$

(d) the derivative of joint cdf with respect to y is

$$\frac{\partial F(x,y)}{\partial y} = \exp(-x) + \left[-1 + \frac{\alpha x}{x+y} - \frac{\alpha xy}{(x+y)^2}\right] \exp\left[\frac{\alpha xy}{x+y} - x - y\right],$$

so the conditional cdf if X given $Y = y$ is

$$F(x|y) = 1 + \left[-1 + \frac{\alpha x}{x+y} - \frac{\alpha xy}{(x+y)^2}\right] \exp\left[\frac{\alpha xy}{x+y} - x\right].$$

(e) the derivative of joint cdf with respect to x and y is

$$\begin{aligned} f(x,y) &= \frac{\partial F(x,y)}{\partial x \partial y} \\ &= \left\{\frac{2\alpha xy}{(x+y)^3} + \left[-1 + \frac{\alpha x}{x+y} - \frac{\alpha xy}{(x+y)^2}\right]\left[-1 + \frac{\alpha y}{x+y} - \frac{\alpha xy}{(x+y)^2}\right]\right\} \\ &\quad \times \exp\left[\frac{\alpha xy}{x+y} - x - y\right]. \end{aligned}$$

(f) the conditional pdf of Y given $X = x$ is

$$\begin{aligned} f(y|x) &= \left\{\frac{2\alpha xy}{(x+y)^3} + \left[-1 + \frac{\alpha x}{x+y} - \frac{\alpha xy}{(x+y)^2}\right]\left[-1 + \frac{\alpha y}{x+y} - \frac{\alpha xy}{(x+y)^2}\right]\right\} \\ &\quad \times \exp\left[\frac{\alpha xy}{x+y} - y\right]. \end{aligned}$$

(g) the conditional pdf of X given $Y = y$ is

$$\begin{aligned} f(x|y) &= \left\{\frac{2\alpha xy}{(x+y)^3} + \left[-1 + \frac{\alpha x}{x+y} - \frac{\alpha xy}{(x+y)^2}\right]\left[-1 + \frac{\alpha y}{x+y} - \frac{\alpha xy}{(x+y)^2}\right]\right\} \\ &\quad \times \exp\left[\frac{\alpha xy}{x+y} - x\right]. \end{aligned}$$

Solutions to question 3:

(a) We can write

$$\overline{F}(x,y) = \exp\left[-\left(x^a + y^a\right)^{1/a}\right] = \exp\left\{-(x+y)\left[\left(\frac{y}{x+y}\right)^a + \left(\frac{x}{x+y}\right)^a\right]\right\}.$$

This is in the form of

$$\overline{F}(x,y) = \exp\left[-(x+y)A\left(\frac{y}{x+y}\right)\right]$$

with $A(t) = [t^a + (1-t)^a]^{1/a}$.

We now check the conditions for $A(\cdot)$. Clearly, $A(0) = 1$ and $A(1) = 1$.

Also $A(t) \geq 0$ since $t^a \geq 0$ and $(1-t)^a \geq 0$ for all t.

To show that $A(t) \leq 1$, note that

$$\begin{aligned} & A(t) \leq 1 \\ \Leftrightarrow\ & [t^a + (1-t)^a]^{1/a} \leq 1 \\ \Leftrightarrow\ & t^a + (1-t)^a \leq 1. \end{aligned}$$

Now let $g(t) = t^a + (1-t)^a$. We have $g^{'}(t) = at^{a-1} - a(1-t)^{a-1}$, $g^{'}(0) = -a$, $g^{'}(1) = a$ and $g^{'}(t) = a(a-1)t^{a-2} + a(a-1)(1-t)^{a-2}$. So, $g(t)$ attains its global maximum at $t = 1/2$ with $g(1/2) = 2^{1-a} \leq 1$. Hence, $t^a + (1-t)^a \leq 1$ holds for all t.

Also $A(t) \geq t$ since

$$[t^a + (1-t)^a]^{1/a} \geq [t^a]^{1/a} \geq t.$$

Also $A(t) \geq 1 - t$ since

$$[t^a + (1-t)^a]^{1/a} \geq [(1-t)^a]^{1/a} \geq 1 - t.$$

$A(\cdot)$ is convex since

$$A^{'}(t) = [t^a + (1-t)^a]^{1/a-1}\left[t^{a-1} - (1-t)^{a-1}\right]$$

and

$$A^{''}(t) = (a-1)\left[t^a + (1-t)^a\right]^{1/a-2}\left[t^a(1-t)^{a-2} + t^{a-2}(1-t)^a + 2t^{a-1}(1-t)^{a-1}\right] \geq 0.$$

(b) the joint cdf is

$$F(x,y) = 1 - \exp(-x) - \exp(-y) + \exp\left[-\left(x^a + y^a\right)^{1/a}\right].$$

(c) the derivative of joint cdf with respect to x is

$$\frac{\partial F(x,y)}{\partial x} = \exp(-x) - x^{a-1}\left(x^a + y^a\right)^{1/a-1}\exp\left[-\left(x^a + y^a\right)^{1/a}\right],$$

so the conditional cdf if Y given $X = x$ is

$$F(y|x) = 1 - x^{a-1}\left(x^a + y^a\right)^{1/a-1}\exp\left[x - \left(x^a + y^a\right)^{1/a}\right].$$

(d) the derivative of joint cdf with respect to y is

$$\frac{\partial F(x,y)}{\partial y} = \exp(-y) - y^{a-1}\left(x^a + y^a\right)^{1/a-1}\exp\left[-\left(x^a + y^a\right)^{1/a}\right],$$

so the conditional cdf if X given $Y = y$ is

$$F(x|y) = 1 - y^{a-1}\left(x^a + y^a\right)^{1/a-1}\exp\left[y - \left(x^a + y^a\right)^{1/a}\right].$$

(e) the derivative of joint cdf with respect to x and y is

$$\begin{aligned}
f(x,y) &= \frac{\partial F(x,y)}{\partial x \partial y} \\
&= (xy)^{a-1}\left(x^a+y^a\right)^{1/a-2}\exp\left[-\left(x^a+y^a\right)^{1/a}\right] \\
&\quad \times \left[a-1+\left(x^a+y^a\right)^{1/a}\right].
\end{aligned}$$

(f) the conditional pdf of Y given $X = x$ is

$$\begin{aligned}
f(y|x) &= (xy)^{a-1}\left(x^a+y^a\right)^{1/a-2}\exp\left[x-\left(x^a+y^a\right)^{1/a}\right] \\
&\quad \times \left[a-1+\left(x^a+y^a\right)^{1/a}\right].
\end{aligned}$$

(g) the conditional pdf of X given $Y = y$ is

$$\begin{aligned}
f(x|y) &= (xy)^{a-1}\left(x^a+y^a\right)^{1/a-2}\exp\left[y-\left(x^a+y^a\right)^{1/a}\right] \\
&\quad \times \left[a-1+\left(x^a+y^a\right)^{1/a}\right].
\end{aligned}$$