MATH4/68181: Extreme values and financial risk
Semester 1
Solutions to problem sheet for Week 11

Solutions to question 1:

(a) The pdf is

$$f(x) = (1+\xi x)^{-1/\xi-1} \exp\left\{-(1+\xi x)^{-1/\xi}\right\}.$$

(b) if $\xi > 0$ then the nth moment is

$$\begin{aligned}
E\left(X^n\right) &= \int_{-1/\xi}^{\infty} x^n (1+\xi x)^{-1/\xi-1} \exp\left\{-(1+\xi x)^{-1/\xi}\right\} dx \\
&= \xi^{-n} \int_0^{\infty} \left(y^{-\xi}-1\right)^n \exp(-y) dy \\
&= \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \int_0^{\infty} y^{-\xi(n-k)} \exp(-y) dy \\
&= \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \Gamma\left(1-\xi(n-k)\right).
\end{aligned}$$

if $\xi < 0$ then the nth moment is

$$\begin{aligned}
E\left(X^n\right) &= \int_{-\infty}^{-1/\xi} x^n (1+\xi x)^{-1/\xi-1} \exp\left\{-(1+\xi x)^{-1/\xi}\right\} dx \\
&= \xi^{-n} \int_0^{\infty} \left(y^{-\xi}-1\right)^n \exp(-y) dy \\
&= \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \int_0^{\infty} y^{-\xi(n-k)} \exp(-y) dy \\
&= \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \Gamma\left(1-\xi(n-k)\right).
\end{aligned}$$

if $\xi = 0$ then the nth moment is

$$\begin{aligned}
E\left(X^n\right) &= \int_{-\infty}^{\infty} x^n \exp(-x) \exp\left\{-\exp(-x)\right\} dx \\
&= (-1)^n \int_0^{\infty} (\ln y)^n \exp(-y) dy \\
&= (-1)^n \frac{\partial^n}{\partial a^n} \int_0^{\infty} y^a \exp(-y) dy \Big|_{a=0} \\
&= (-1)^n \frac{\partial^n}{\partial a^n} \Gamma(a+1) \Big|_{a=0}.
\end{aligned}$$

(c) if $\xi \neq 0$ then the mean is

$$E(X) = (1/\xi)\left[\Gamma(1-\xi)-1\right].$$

if $\xi = 0$ then the mean is

$$E(X) = -\frac{\partial}{\partial a}\Gamma(a+1)\bigg|_{a=0} = -\Gamma^{'}(1).$$

(d) if $\xi \neq 0$ then

$$E\left(X^2\right) = \left(1/\xi^2\right)\left[\Gamma(1-2\xi) - 2\Gamma(1-\xi) + 1\right],$$

so

$$Var(X) = \left(1/\xi^2\right)\left[\Gamma(1-2\xi) - \Gamma^2(1-\xi)\right].$$

if $\xi = 0$ then

$$E\left(X^2\right) = \frac{\partial^2}{\partial a^2}\Gamma(a+1)\bigg|_{a=0} = \Gamma^{''}(1),$$

so

$$Var(X) = \Gamma^{''}(1) - \left[\Gamma^{'}(1)\right]^2.$$

Solutions to question 2:

(a) The pdf is

$$f(x) = (1+\xi x)^{-1/\xi-1}.$$

(b) if $\xi > 0$ then the nth moment is

$$\begin{aligned} E(X^n) &= \int_0^\infty x^n (1+\xi x)^{-1/\xi-1} dx \\ &= \xi^{-n-1}\int_1^\infty (y-1)^n y^{-1/\xi-1} dy \\ &= \xi^{-n-1}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\int_1^\infty y^{k-1/\xi-1} dy \\ &= \xi^{-n-1}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{n-k}}{1/\xi - k}. \end{aligned}$$

if $\xi < 0$ then the nth moment is

$$\begin{aligned} E(X^n) &= \int_0^\infty x^n (1+\xi x)^{-1/\xi-1} dx \\ &= \xi^{-n-1}\int_0^1 (y-1)^n y^{-1/\xi-1} dy \\ &= \xi^{-n-1}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\int_0^1 y^{k-1/\xi-1} dy \\ &= \xi^{-n-1}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{n-k}}{1/\xi - k}. \end{aligned}$$

if $\xi = 0$ then the nth moment is

$$E\left(X^n\right) = \int_0^\infty x^n \exp(-x)dx$$
$$= \Gamma(n+1)$$
$$= n!.$$

(c) if $\xi \neq 0$ then the mean is

$$E\left(X\right) = \xi^{-2}\left[-\xi + \frac{1}{1/\xi - 1}\right] = \frac{1}{1-\xi}.$$

if $\xi = 0$ then the mean is

$$E\left(X\right) = 1.$$

(d) if $\xi \neq 0$ then

$$E\left(X^2\right) = \xi^{-3}\left[\xi - \frac{2}{1/\xi - 1} + \frac{1}{1/\xi - 2}\right] = \frac{1}{\xi^2} - \frac{2}{\xi(1-\xi)} + \frac{1}{\xi(1-2\xi)},$$

so

$$Var\left(X\right) = \frac{1}{\xi^2} - \frac{2}{\xi(1-\xi)} + \frac{1}{\xi(1-2\xi)} - \frac{1}{(1-\xi)^2}.$$

if $\xi = 0$ then

$$E\left(X^2\right) = 2! = 2,$$

so

$$Var\left(X\right) = 2 - 1^2 = 1.$$