# MATH4/68181: Extreme values and financial risk
# Semester 1
# Solutions to problem sheet for Week 8

1. From problem sheet 6,

$$\mathrm{VaR}_p(X) = -\frac{1}{\lambda}\log(1-p)$$

and

$$\mathrm{ES}_p(X) = -\frac{1}{p\lambda}\left\{\log(1-p)p - p - \log(1-p)\right\}.$$

If $x_1, x_2, \ldots, x_n$ is a random sample from Exp $(\lambda)$ then the likelihood function is

$$L(\lambda) = \lambda^n \exp\left(-\lambda \sum_{i=1}^n x_i\right).$$

The log-likelihood function is

$$\log L(\lambda) = n \log \lambda - \lambda \sum_{i=1}^n x_i.$$

The derivative with respect to $\lambda$ is

$$\frac{d \log L(\lambda)}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^n x_i.$$

Setting this to zero gives the mle of $\lambda$ as

$$\widehat{\lambda} = \frac{n}{\sum_{i=1}^n x_i}.$$

Hence, the mles of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$ are

$$\widehat{\mathrm{VaR}}_p(X) = -\frac{\sum_{i=1}^n x_i}{n}\log(1-p)$$

and

$$\widehat{\mathrm{ES}}_p(X) = -\frac{\sum_{i=1}^n x_i}{pn}\left\{\log(1-p)p - p - \log(1-p)\right\}.$$

2. From problem sheet 6,

$$\mathrm{VaR}_p(X) = p^{1/a}$$

and

$$\mathrm{ES}_p(X) = \frac{p^{1/a}}{1/a+1}.$$

If $x_1, x_2, \ldots, x_n$ is a random sample from the power function distribution stated in the question then the likelihood function is

$$L(a) = a^n \left( \prod_{i=1}^{n} x_i \right)^{a-1}.$$

The log-likelihood function is

$$\log L(a) = n \log a + (a-1) \sum_{i=1}^{n} \log x_i.$$

The derivative with respect to $a$ is

$$\frac{d \log L(a)}{da} = \frac{n}{a} + \sum_{i=1}^{n} \log x_i.$$

Setting this to zero gives the mle of $a$ as

$$\widehat{a} = -\frac{n}{\sum\limits_{i=1}^{n} \log x_i}.$$

Hence, the mles of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$ are

$$\widehat{\mathrm{VaR}}_p(X) = p^{-\sum_{i=1}^{n} \log x_i / n}$$

and

$$\widehat{\mathrm{ES}}_p(X) = \frac{p^{-\sum_{i=1}^{n} \log x_i / n}}{-\sum\limits_{i=1}^{n} \log x_i / n + 1}.$$

3. Setting

$$\Phi\left(\frac{x-\mu}{\sigma}\right) = p$$

gives

$$\mathrm{VaR}_p(X) = \mu + \sigma \Phi^{-1}(p).$$

So,

$$\mathrm{ES}_p(X) = \frac{1}{p} \int_0^p \left[ \mu + \sigma \Phi^{-1}(v) \right] dv = \mu + \frac{\sigma}{p} \int_0^p \Phi^{-1}(v) dv.$$

If $x_1, x_2, \ldots, x_n$ is a random sample from $N(\mu, \sigma^2)$ then the likelihood function is

$$L(\mu, \sigma) = \frac{1}{(2\pi)^{n/2}\sigma^n} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 \right\}.$$

The log-likelihood function is

$$\log L(\mu, \sigma) = -\frac{n}{2} \log(2\pi) - n \log \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 .$$

The partial derivatives with respect to $\mu$ and $\sigma$ are

$$\frac{\partial \log L}{\partial \mu} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu)$$

and

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} (x_i - \mu)^2 .$$

Setting the first partial derivative to zero, we obtain the mle of $\mu$ as

$$\widehat{\mu} = \frac{1}{n} \sum_{i=1}^{n} x_i = \overline{x}.$$

Setting the second partial derivative to zero, we obtain the mle of $\sigma^2$ as

$$\widehat{\sigma^2} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \widehat{\mu})^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \overline{x})^2 .$$

Hence, the mles of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$ are

$$\widehat{\mathrm{VaR}}_p(X) = \overline{x} + \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \overline{x})^2} \Phi^{-1}(p)$$

and

$$\widehat{\mathrm{ES}}_p(X) = \overline{x} + \frac{1}{p} \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \overline{x})^2} \int_0^p \Phi^{-1}(v) dv.$$

4. Setting

$$\Phi \left( \frac{\log x - \mu}{\sigma} \right) = p$$

gives

$$\mathrm{VaR}_p(X) = \exp\left[ \mu + \sigma \Phi^{-1}(p) \right].$$

So,

$$\mathrm{ES}_p(X) = \frac{1}{p} \int_0^p \exp\left[ \mu + \sigma \Phi^{-1}(v) \right] dv = \frac{1}{p} \exp(\mu) \int_0^p \exp\left[ \sigma \Phi^{-1}(v) \right] dv.$$

If $x_1, x_2, \ldots, x_n$ is a random sample from $LN(\mu, \sigma^2)$ then the likelihood function is

$$L(\mu, \sigma) = \frac{1}{(2\pi)^{n/2}\sigma^n} \left( \prod_{i=1}^{n} x_i \right)^{-1} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (\log x_i - \mu)^2 \right\}.$$

The log-likelihood function is

$$\log L(\mu, \sigma) = -\frac{n}{2} \log(2\pi) - n \log \sigma - \sum_{i=1}^{n} \log x_i - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (\log x_i - \mu)^2 .$$

The partial derivatives with respect to $\mu$ and $\sigma$ are

$$\frac{\partial \log L}{\partial \mu} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (\log x_i - \mu)$$

and

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} (\log x_i - \mu)^2 .$$

Setting the first partial derivative to zero, we obtain the mle of $\mu$ as

$$\widehat{\mu} = \frac{1}{n} \sum_{i=1}^{n} \log x_i.$$

Setting the second partial derivative to zero, we obtain the mle of $\sigma^2$ as

$$\widehat{\sigma^2} = \frac{1}{n} \sum_{i=1}^{n} (\log x_i - \widehat{\mu})^2 = \frac{1}{n} \sum_{i=1}^{n} \left( \log x_i - \frac{1}{n} \sum_{j=1}^{n} \log x_j \right)^2 .$$

Hence, the mles of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$ are

$$\mathrm{VaR}_p(X) = \exp \left[ \frac{1}{n} \sum_{i=1}^{n} \log x_i + \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left( \log x_i - \frac{1}{n} \sum_{j=1}^{n} \log x_j \right)^2} \Phi^{-1}(p) \right]$$

and

$$\mathrm{ES}_p(X) = \frac{1}{p} \exp \left( \frac{1}{n} \sum_{i=1}^{n} \log x_i \right) \int_0^p \exp \left[ \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left( \log x_i - \frac{1}{n} \sum_{j=1}^{n} \log x_j \right)^2} \Phi^{-1}(v) \right] dv.$$

5. The cdf corresponding to the given pdf is

$$F(x) = \theta_2 \theta_1^{-\theta_2} \int_0^x y^{\theta_2 - 1} dy = \theta_1^{-\theta_2} x^{\theta_2}.$$

Setting

$$\theta_1^{-\theta_2} x^{\theta_2} = p$$

gives

$$\mathrm{VaR}_p(X) = \theta_1 p^{1/\theta_2}.$$

So,

$$\mathrm{ES}_p(X) = \frac{1}{p}\int_0^p \theta_1 v^{1/\theta_2} dv = \frac{\theta_1\theta_2}{1+\theta_2} p^{1/\theta_2}.$$

The joint likelihood function of $\theta_1$ and $\theta_2$ is

$$L\left(\theta_1, \theta_2\right) = \theta_2^n \theta_1^{-n\theta_2} \left(\prod_{i=1}^n x_i\right)^{\theta_2 - 1}$$

for $\theta_1 > 0$ and $\theta_2 > 0$.

The likelihood function monotonically decreases with respect to $\theta_1$. The lowest possible value for $\theta_1$ is $\max(x_1, x_2, \ldots, x_n)$. So, the mle of $\theta_1$ is $\max(x_1, x_2, \ldots, x_n)$.

The log of the joint likelihood function is

$$\log L\left(\theta_1, \theta_2\right) = n \log \theta_2 - n\theta_2 \log \theta_1 + \left(\theta_2 - 1\right) \sum_{i=1}^n \log x_i.$$

The first derivative of the log likelihood with respect to $\theta_2$ is

$$\frac{d \log L\left(\theta_1, \theta_2\right)}{d\theta_2} = \frac{n}{\theta_2} - n \log \theta_1 + \sum_{i=1}^n \log x_i.$$

Setting this to zero and solving, we obtain $\widehat{\theta_2} = n / \left\{ n \log \widehat{\theta_1} - \sum_{i=1}^n \log x_i \right\}$, where $\widehat{\theta_1} = \max(x_1, x_2, \ldots, x_n)$. The second derivative of the log likelihood with respect to $\theta_2$

$$\frac{d^2 \log L\left(\theta_1, \theta_2\right)}{d\theta_2^2} = -\frac{n}{\theta_2^2} < 0.$$

So, $\widehat{\theta_2} = n / \left\{ n \log \widehat{\theta_1} - \sum_{i=1}^n \log x_i \right\}$ is indeed the mle of $\theta_2$.

Hence, the mles of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$ are

$$\widehat{\mathrm{VaR}}_p(X) = \max\left(x_1, x_2, \ldots, x_n\right) p^{\log \max(x_1, x_2, \ldots, x_n) - (1/n) \sum_{i=1}^n \log x_i}$$

and

$$\widehat{\mathrm{ES}}_p(X) = \frac{\max\left(x_1, x_2, \ldots, x_n\right)}{\log \max\left(x_1, x_2, \ldots, x_n\right) - (1/n) \sum_{i=1}^n \log x_i + 1} p^{\log \max(x_1, x_2, \ldots, x_n) - (1/n) \sum_{i=1}^n \log x_i}.$$

6. The cdf corresponding to the given pdf is

$$F(x) = \frac{x - \mu + \delta}{2\delta}.$$

Setting

$$\frac{x - \mu + \delta}{2\delta} = p$$

gives

$$\mathrm{VaR}_p(X) = \mu - \delta + 2\delta p.$$

So,

$$\mathrm{ES}_p(X) = \frac{1}{p}\int_0^p [\mu - \delta + 2\delta v]\, dv = \mu - \delta + \delta p.$$

The joint likelihood function of $\mu$ and $\delta$ is

$$\begin{aligned} L(\mu, \delta) &= (2\delta)^{-n} \prod_{i=1}^{n} I\left\{\mu - \delta < x_i < \mu + \delta\right\} \\ &= (2\delta)^{-n} \prod_{i=1}^{n} I\left\{\max\left(x_1, x_2, \ldots, x_n\right) < \mu + \delta, \min\left(x_1, x_2, \ldots, x_n\right) > \mu - \delta\right\} \\ &= (2\delta)^{-n} \prod_{i=1}^{n} I\left\{\max\left(x_1, x_2, \ldots, x_n\right) - \delta < \mu < \min\left(x_1, x_2, \ldots, x_n\right) + \delta\right\}. \end{aligned} \tag{1}$$

For (1) to be valid, we must have $\max(x_1, x_2, \ldots, x_n) - \delta < \min(x_1, x_2, \ldots, x_n) + \delta$. In other words, $\delta \geq (1/2)\{\max(x_1, x_2, \ldots, x_n) - \min(x_1, x_2, \ldots, x_n)\}$. Since (1) is a decreasing function of $\delta$, it follows that $\widehat{\delta} = (1/2)\{\max(x_1, x_2, \ldots, x_n) - \min(x_1, x_2, \ldots, x_n)\}$.

Substituting the solution $\widehat{\delta} = (1/2)\{\max(x_1, x_2, \ldots, x_n) - \min(x_1, x_2, \ldots, x_n)\}$ into (1), we see that the only possible value for $\mu$ is $\widehat{\mu} = (1/2)\{\min(x_1, x_2, \ldots, x_n) + \max(x_1, x_2, \ldots, x_n)\}$.

Hence, the mles of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$ are

$$\widehat{\mathrm{VaR}}_p(X) = \min\left(x_1, x_2, \ldots, x_n\right) + \left\{\max\left(x_1, x_2, \ldots, x_n\right) - \min\left(x_1, x_2, \ldots, x_n\right)\right\} p$$

and

$$\widehat{\mathrm{ES}}_p(X) = \min\left(x_1, x_2, \ldots, x_n\right) + (1/2)\left\{\max\left(x_1, x_2, \ldots, x_n\right) - \min\left(x_1, x_2, \ldots, x_n\right)\right\} p.$$