MATH4/68181: Extreme values and financial risk
Semester 1
Solutions to problem sheet for Week 2

1. The density function of the Gumbel extreme value distribution is $\exp(-x)\exp\{-\exp(-x)\}$. The density function of the Fréchet extreme value distribution is $\alpha x^{-\alpha-1}\exp\{-x^{-\alpha}\}$. The density function of the Weibull extreme value distribution is $\alpha(-x)^{\alpha-1}\exp\{-(-x)^{\alpha}\}$.

2. The mean of the Gumbel extreme value distribution is

$$
\begin{aligned}
\int_{-\infty}^{\infty} x\exp(-x)\exp\left\{-\exp(-x)\right\}dx &= -\int_0^{\infty}\ln y\exp\left\{-y\right\}dy \quad [y=\exp(-x)] \\
&= -\frac{d}{d\alpha}\int_0^{\infty} y^{\alpha}\exp\left\{-y\right\}dy\Big|_{\alpha=0} \\
&= -\frac{d}{d\alpha}\Gamma(\alpha+1)\Big|_{\alpha=0} \\
&= -\Gamma^{'}(1).
\end{aligned}
$$

The mean of the Fréchet extreme value distribution is

$$
\begin{aligned}
\int_0^{\infty} x\alpha x^{-\alpha-1}\exp\left\{-x^{-\alpha}\right\}dx &= \alpha\int_0^{\infty} x^{-\alpha}\exp\left\{-x^{-\alpha}\right\}dx \\
&= \alpha\int_0^{\infty} x^{-\alpha}\exp\left\{-x^{-\alpha}\right\}dx \\
&= \alpha\int_0^{\infty} y\exp\left\{-y\right\}dx \quad [y=x^{-\alpha}] \\
&= \alpha\int_{\infty}^{0} y\exp\left\{-y\right\}\frac{dx}{dy}dy \\
&= \alpha\int_{\infty}^{0} y\exp\left\{-y\right\}\frac{dy^{-1/\alpha}}{dy}dy \\
&= -\int_{\infty}^{0} y\exp\left\{-y\right\}y^{-1-\frac{1}{\alpha}}dy \\
&= -\int_{\infty}^{0} y^{-\frac{1}{\alpha}}\exp\left\{-y\right\}dy \\
&= \int_0^{\infty} y^{-\frac{1}{\alpha}}\exp\left\{-y\right\}dy \\
&= \Gamma\left(1-\frac{1}{\alpha}\right).
\end{aligned}
$$

The mean of the Weibull extreme value distribution is

$$
\begin{aligned}
\int_{-\infty}^{0} x\alpha(-x)^{\alpha-1}\exp\left\{-(-x)^{\alpha}\right\}dx &= -\alpha\int_{-\infty}^{0}(-x)^{\alpha}\exp\left\{-(-x)^{\alpha}\right\}dx \\
&= -\alpha\int_{-\infty}^{0} y\exp\left\{-y\right\}dx \quad [y=(-x)^{\alpha}] \\
&= -\alpha\int_{\infty}^{0} y\exp\left\{-y\right\}\frac{dx}{dy}dy
\end{aligned}
$$

$$
\begin{aligned}
&= -\alpha \int_{\infty}^{0} y \exp\{-y\} \frac{d\left(-y^{1/\alpha}\right)}{dy} dy \\
&= \int_{\infty}^{0} y \exp\{-y\} y^{\frac{1}{\alpha}-1} dy \\
&= \int_{\infty}^{0} y^{\frac{1}{\alpha}} \exp\{-y\} dy \\
&= -\int_{0}^{\infty} y^{\frac{1}{\alpha}} \exp\{-y\} dy \\
&= -\Gamma\left(\frac{1}{\alpha}+1\right).
\end{aligned}
$$

3. Since

$$
\begin{aligned}
\int_{-\infty}^{\infty} x^2 \exp(-x) \exp\{-\exp(-x)\} dx &= \int_0^{\infty} (\ln y)^2 \exp\{-y\} dy \quad [y=\exp(-x)] \\
&= \left. \frac{d^2}{d\alpha^2} \int_0^{\infty} y^{\alpha} \exp\{-y\} dy \right|_{\alpha=0} \\
&= \left. \frac{d^2}{d\alpha^2} \Gamma(\alpha+1) \right|_{\alpha=0} \\
&= \Gamma^{''}(1),
\end{aligned}
$$

the variance of the Gumbel extreme value distribution is $\Gamma^{''}(1) - [\Gamma^{'}(1)]^2$. Since

$$
\begin{aligned}
\int_0^{\infty} x^2 \alpha x^{-\alpha-1} \exp\{-x^{-\alpha}\} dx &= \int_0^{\infty} y^{-2/\alpha} \exp\{-y\} dy \quad [y=\exp(-x)] \\
&= \Gamma(1-2/\alpha),
\end{aligned}
$$

the variance of the Fréchet extreme value distribution is $\Gamma(1-2/\alpha) - \Gamma^2(1-1/\alpha)$. Since

$$
\begin{aligned}
\int_{-\infty}^{0} x^2 \alpha (-x)^{\alpha-1} \exp\{-(-x)^{\alpha}\} dx &= \int_0^{\infty} y^{2/\alpha} \exp\{-y\} dy \quad [y=(-x)^{\alpha}] \\
&= \Gamma(2/\alpha+1),
\end{aligned}
$$

the variance of the Weibull extreme value distribution is $\Gamma(2/\alpha+1) - \Gamma^2(1/\alpha+1)$.

4. We have

$$
\begin{aligned}
& \Lambda^n(x) = \Lambda(\alpha_n x + \beta_n) \\
\Leftrightarrow \quad & \exp\{-n\exp(-x)\} = \exp\{-\exp(-\alpha_n x - \beta_n)\} \\
\Leftrightarrow \quad & n\exp(-x) = \exp(-\alpha_n x - \beta_n) \\
\Leftrightarrow \quad & \log n - x = -\alpha_n x - \beta_n.
\end{aligned}
$$

Hence, the result.

5. We have

$$
\begin{aligned}
& \Phi_{\alpha}^n(x) = \Phi_{\alpha}(\alpha_n x + \beta_n) \\
\Leftrightarrow \quad & \exp\left(-n x^{-\alpha}\right) = \exp\left(-(\alpha_n x + \beta_n)^{-\alpha}\right) \\
\Leftrightarrow \quad & n x^{-\alpha} = (\alpha_n x + \beta_n)^{-\alpha} \\
\Leftrightarrow \quad & n^{-1/\alpha} x = \alpha_n x + \beta_n.
\end{aligned}
$$

Hence, the result.

6. We have

$$\begin{aligned} & \Psi_\alpha^n(x) = \Psi_\alpha\left(\alpha_n x + \beta_n\right) \\ \Leftrightarrow \quad & \exp\left(-n(-x)^\alpha\right) = \exp\left(-\left(-\alpha_n x - \beta_n\right)^\alpha\right) \\ \Leftrightarrow \quad & n(-x)^\alpha = \left(-\alpha_n x - \beta_n\right)^\alpha \\ \Leftrightarrow \quad & n^{1/\alpha} x = \alpha_n x + \beta_n. \end{aligned}$$

Hence, the result.

7. Note that $w(F) = \infty$ and take $\gamma(t) = 1$. Then

$$\lim_{t \uparrow \infty} \frac{1 - F\left(t + x\right)}{1 - F(t)} = \frac{\exp\left(-t - x\right)}{\exp(-t)} = \exp(-x).$$

. So, the exponential cdf $F(x) = 1 - \exp(-x)$ belongs to the Gumbel domain of attraction.

8. Note that $w(F) = \infty$ and take $\gamma(t) = 1$. Then

$$\lim_{t \uparrow \infty} \frac{1 - F\left(t + x\right)}{1 - F(t)} = \lim_{t \uparrow \infty} \frac{1 - \left[1 - \exp\left(-t - x\right)\right]^\alpha}{1 - \left[1 - \exp\left(-t\right)\right]^\alpha} = \lim_{t \uparrow \infty} \frac{\alpha \exp\left(-t - x\right)}{\alpha \exp\left(-t\right)} = \exp(-x).$$

. So, the exponentiated exponential cdf $F(x) = [1 - \exp(-x)]^\alpha$ belongs to the Gumbel domain of attraction.

9. Note that $w(F) = 1$. Then

$$\lim_{t \downarrow 0} \frac{1 - F\left(1 - tx\right)}{1 - F(1 - t)} = \frac{1 - (1 - tx)}{1 - (1 - t)} = x.$$

. So, the uniform$[0, 1]$ cdf $F(x) = x$ belongs to the Weibull domain of attraction.

10. We have

$$\lim_{t \uparrow \infty} \frac{1 - F\left(tx\right)}{1 - F(t)} = \frac{(K/(tx))^\alpha}{(K/t)^\alpha} = x^{-\alpha}.$$

. So, the Pareto cdf $F(x) = 1 - (K/x)^\alpha$ belongs to the Fréchet domain of attraction.

11. Firstly, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function, say $h(t)$, such that

$$\lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} = \exp(-x)$$

for every $x \in (-\infty, \infty)$. But, using L'Hopital's rule, we note that

$$\begin{aligned} \lim_{t \to w(F)} \frac{1 - F\left(t + x\,h(t)\right)}{1 - F(t)} &= \lim_{t \to w(F)} \frac{\left[1 + xh^{'}(t)\right] f\left(t + x\,h(t)\right)}{f(t)} \\ &= \lim_{t \to w(G)} \frac{\left[1 + xh^{'}(t)\right] g\left(t + xh(t)\right)}{g(t)} \left[\frac{G\left(t + xh(t)\right)}{G(t)}\right]^{a-1} \\ & \quad \times \left[\frac{1 - G\left(t + xh(t)\right)}{1 - G(t)}\right]^{b-1} \exp\left\{cG(t) - cG\left(t + xh(t)\right)\right\} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{t \to w(G)} \frac{\left[1 + xh^{'}(t)\right] g\left(t + xh(t)\right)}{g(t)} \left[\frac{1}{1}\right]^{a-1} \\
&\quad \times \left[\frac{1 - G\left(t + xh(t)\right)}{1 - G(t)}\right]^{b-1} \exp\left\{c - c\right\} \\
&= \lim_{t \to w(G)} \frac{\left[1 + xh^{'}(t)\right] g\left(t + xh(t)\right)}{g(t)} \\
&\quad \times \left[\frac{1 - G\left(t + xh(t)\right)}{1 - G(t)}\right]^{b-1} \\
&= \lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} \\
&\quad \times \left[\frac{1 - G\left(t + xh(t)\right)}{1 - G(t)}\right]^{b-1} \\
&= \left[\frac{1 - G\left(t + xh(t)\right)}{1 - G(t)}\right]^{b} \\
&= \left[\exp(-x)\right]^{b} \\
&= \exp(-bx)
\end{aligned}
$$

for every $x \in (-\infty, \infty)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$
\lim_{n \to \infty} \Pr\left\{a_n \left(M_n - b_n\right) \leq x\right\} = \exp\left\{-\exp(-b\, x)\right\}
$$

for some suitable norming constants $a_n > 0$ and b_n.

Secondly, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta < 0$ such that

$$
\lim_{t \to \infty} \frac{1 - G(t\, x)}{1 - G(t)} = x^{\beta}
$$

for every $x > 0$. But, using L'Hopital's rule, we note that

$$
\begin{aligned}
\lim_{t \to \infty} \frac{1 - F(t\, x)}{1 - F(t)} &= \lim_{t \to \infty} \frac{x f(t\, x)}{f(t)} \\
&= \lim_{t \to \infty} \frac{x\, g(t\, x)}{g(t)} \left[\frac{G(t\, x)}{G(t)}\right]^{a-1} \left[\frac{1 - G(t\, x)}{1 - G(t)}\right]^{b-1} \exp\left\{c\, G(t) - c\, G\left(t\, x\right)\right\} \\
&= \lim_{t \to \infty} \frac{x\, g(t\, x)}{g(t)} \left[\frac{1 - G(t\, x)}{1 - G(t)}\right]^{b-1} \\
&= \lim_{t \to \infty} \frac{1 - G(t\, x)}{1 - G(t)} \left[\frac{1 - G(t\, x)}{1 - G(t)}\right]^{b-1} \\
&= \lim_{t \to \infty} \left[\frac{1 - G(t\, x)}{1 - G(t)}\right]^{b} \\
&= x^{b\, \beta}
\end{aligned}
$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$
\lim_{n \to \infty} \Pr\left\{a_n \left(M_n - b_n\right) \leq x\right\} = \exp\left(-x^{b\beta}\right)
$$

for some suitable norming constants $a_n > 0$ and b_n.

Thirdly, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\alpha > 0$ such that

$$\lim_{t \to 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} = x^{\alpha}$$

for every $x > 0$. But, using L'Hopital's rule, we note that

$$\begin{aligned}
\lim_{t \to 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} &= \lim_{t \to 0} \frac{x f(w(F) - tx)}{f(w(F) - t)} \\
&= \lim_{t \to 0} \frac{x g(w(F) - tx)}{g(w(F) - t)} \left[\frac{G(w(F) - tx)}{G(w(F) - t)} \right]^{a-1} \left[\frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \right]^{b-1} \\
&\quad \times \exp \left\{ cG(w(F) - t) - cG\left(w(F) - tx\right) \right\} \\
&= \lim_{t \to 0} \frac{x g(w(F) - tx)}{g(w(F) - t)} \left[\frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \right]^{b-1} \\
&= \lim_{t \to 0} \frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \left[\frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \right]^{b-1} \\
&= \lim_{t \to 0} \left[\frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \right]^{b} \\
&= x^{b\alpha}.
\end{aligned}$$

So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{n \to \infty} \Pr \left\{ a_n \left(M_n - b_n \right) \leq x \right\} = \exp \left\{ -(-x)^{b\alpha} \right\}$$

for some suitable norming constants $a_n > 0$ and b_n.

12. First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function, say $h(t)$, such that

$$\lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} = \exp(-x)$$

for every $x \in (-\infty, \infty)$. But, using l'Hopital's rule, we note that

$$\begin{aligned}
\lim_{t \to w(F)} \frac{1 - F_X\left(t + xh(t)\right)}{1 - F_X(t)} &= \lim_{t \to w(F)} \frac{\left(1 + xh^{'}(t)\right) f_X\left(t + xh(t)\right)}{f_X(t)} \\
&= \lim_{t \to w(G)} \frac{\left(1 + xh^{'}(t)\right) g\left(t + xh(t)\right)}{g(t)} \left[\frac{G\left(t + xh(t)\right)}{G(t)} \right]^{a-1} \\
&\quad \times \left[\frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} \right]^{b-1} \left[\frac{1 - (1 - \beta) G\left(t\right)}{1 - (1 - \beta) G\left(t + xh(t)\right)} \right]^{a+b} \\
&= \lim_{t \to w(G)} \frac{\left(1 + xh^{'}(t)\right) g\left(t + xh(t)\right)}{g(t)} \left[\frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} \right]^{b-1} \\
&= \exp(-bx)
\end{aligned}$$

for every $x \in (-\infty, \infty)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n \to \infty} \Pr \left\{ a_n \left(M_n - b_n \right) \leq x \right\} = \exp \left\{ -\exp(-bx) \right\}$$

for some suitable norming constants $a_n > 0$ and b_n.

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $c < 0$, such that

$$\lim_{t \to \infty} \frac{1 - G(tx)}{1 - G(t)} = x^c$$

for every $x > 0$. But, using l'Hopital's rule, we note that

$$\begin{aligned}
\lim_{t \to \infty} \frac{1 - F_X\left(tx\right)}{1 - F_X(t)} &= \lim_{t \to \infty} \frac{x f_X(tx)}{f_X(t)} \\
&= \lim_{t \to \infty} \frac{xg\left(tx\right)}{g(t)} \left[\frac{G\left(tx\right)}{G(t)} \right]^{a-1} \left[\frac{1 - G\left(tx\right)}{1 - G(t)} \right]^{b-1} \left[\frac{1 - (1-\beta)G\left(t\right)}{1 - (1-\beta)G\left(tx\right)} \right]^{a+b} \\
&= \lim_{t \to \infty} \frac{xg\left(tx\right)}{g(t)} \left[\frac{1 - G\left(tx\right)}{1 - G(t)} \right]^{b-1} \\
&= \lim_{t \to \infty} \frac{1 - G\left(tx\right)}{1 - G(t)} \left[\frac{1 - G\left(tx\right)}{1 - G(t)} \right]^{b-1} \\
&= \lim_{t \to \infty} \left[\frac{1 - G\left(tx\right)}{1 - G(t)} \right]^{b} \\
&= x^{bc}
\end{aligned}$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n \to \infty} \Pr \left\{ a_n \left(M_n - b_n \right) \leq x \right\} = \exp \left(-x^{bc} \right)$$

for some suitable norming constants $a_n > 0$ and b_n.

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $c > 0$, such that

$$\lim_{t \to 0} \frac{1 - G(w(G) - tx)}{G(w(G) - t)} = x^c$$

for every $x < 0$. But, using l'Hopital's rule and results in Section 2.2, we note that

$$\begin{aligned}
\lim_{t \to 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F(w(F) - t)} &= \lim_{t \to 0} \frac{x f(w(F) - tx)}{f(w(F) - t)} \\
&= \lim_{t \to 0} \frac{xg\left(w(F) - tx\right)}{g(w(F) - t)} \left[\frac{G\left(w(F) - tx\right)}{G(w(F) - t)} \right]^{a-1} \left[\frac{1 - G\left(w(F) - tx\right)}{1 - G(w(F) - t)} \right]^{b-1} \\
&\quad \times \left[\frac{1 - (1-\beta)G\left(w(F) - t\right)}{1 - (1-\beta)G\left(w(F) - tx\right)} \right]^{a+b} \\
&= \lim_{t \to 0} \frac{xg\left(w(F) - tx\right)}{g(w(F) - t)} \left[\frac{1 - G\left(w(F) - tx\right)}{1 - G(w(F) - t)} \right]^{b-1}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{t \to 0} \frac{1 - G\left(w(F) - tx\right)}{1 - G(w(F) - t)} \left[\frac{1 - G\left(w(F) - tx\right)}{1 - G(w(F) - t)}\right]^{b-1} \\
&= \lim_{t \to 0} \left[\frac{1 - G\left(w(F) - tx\right)}{1 - G(w(F) - t)}\right]^{b} \\
&= x^{bc}
\end{aligned}
$$

for every $x < 0$. So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$
\lim_{n \to \infty} \Pr\left\{a_n \left(M_n - b_n\right) \le x\right\} = \exp\left\{-(-x)^{bc}\right\}
$$

for some suitable norming constants $a_n > 0$ and b_n.

13. We have

$$
\begin{aligned}
& G(x) = \exp\left\{-\left(1 + \xi \frac{x - \mu}{\sigma}\right)^{-1/\xi}\right\} = p \\
\Leftrightarrow \quad & \left(1 + \xi \frac{x - \mu}{\sigma}\right)^{-1/\xi} = -\log p \\
\Leftrightarrow \quad & 1 + \xi \frac{x - \mu}{\sigma} = (-\log p)^{-\xi} \\
\Leftrightarrow \quad & \xi \frac{x - \mu}{\sigma} = (-\log p)^{-\xi} - 1 \\
\Leftrightarrow \quad & x = \mu + \frac{\sigma}{\xi}\left[(-\log p)^{-\xi} - 1\right].
\end{aligned}
$$

Hence, the result.

14. We have

$$
\begin{aligned}
& G(x) = 1 - \left\{1 + \xi \frac{x - t}{\sigma}\right\}^{-1/\xi} = p \\
\Leftrightarrow \quad & 1 + \xi \frac{x - t}{\sigma} = (1 - p)^{-\xi} \\
\Leftrightarrow \quad & \xi \frac{x - t}{\sigma} = (1 - p)^{-\xi} - 1 \\
\Leftrightarrow \quad & x = t + \frac{\sigma}{\xi}\left[(1 - p)^{-\xi} - 1\right].
\end{aligned}
$$

Hence, the result.