MATH48181/68181: EXTREME VALUES
FIRST SEMESTER
ANSWERS TO IN CLASS TEST

ANSWER TO QUESTION 1 If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cumulative distribution function of a normalized version of M_n converges to G, i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n\left(a_n x + b_n\right) \rightarrow G(x) \tag{1}$$

as $n \rightarrow \infty$ then G must be of the same type as (cumulative distribution functions G and G^* are of the same type if $G^*(x) = G(ax + b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$\begin{aligned}
I &: \Lambda(x) = \exp\left\{-\exp(-x)\right\}, \qquad x \in \Re; \\
II &: \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\left\{-x^{-\alpha}\right\} & \text{if } x \geq 0 \end{cases} \\
& \quad \text{for some } \alpha > 0; \\
III &: \Psi_\alpha(x) = \begin{cases} \exp\left\{-(-x)^{\alpha}\right\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \\
& \quad \text{for some } \alpha > 0.
\end{aligned}$$

(1 marks)

The necessary and sufficient conditions for the three extreme value distributions are:

$$\begin{aligned}
I &: \exists \gamma(t) > 0 \text{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F\left(t + x\gamma(t)\right)}{1 - F(t)} = \exp(-x), \qquad x > 0, \\
II &: w(F) = \infty \text{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \qquad x > 0, \\
III &: w(F) < \infty \text{ and } \lim_{t \downarrow 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} = x^{\alpha}, \qquad x > 0.
\end{aligned}$$

(1 marks)

Firstly, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function, say $h(t)$, such that

$$\lim_{t \rightarrow w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} = \exp(-x)$$

for every $x > 0$. But,

$$\lim_{t \rightarrow w(F)} \frac{1 - F\left(t + x\ h(t)\right)}{1 - F(t)} = \lim_{t \rightarrow w(F)} \frac{1 - \dfrac{G\left(t + x\ h(t)\right)}{a + (1-a)G\left(t + x\ h(t)\right)}}{1 - \dfrac{G(t)}{a + (1-a)G(t)}}$$

$$
\begin{aligned}
&= \lim_{t \to w(G)} \frac{1 - \dfrac{G(t + x\ h(t))}{a + (1-a)G(t + x\ h(t))}}{1 - \dfrac{G(t)}{a + (1-a)G(t)}} \\
&= \lim_{t \to w(G)} \frac{1 - \dfrac{G(t + x\ h(t))}{a + 1 - a}}{1 - \dfrac{G(t)}{a + 1 - a}} \\
&= \lim_{t \to w(G)} \frac{1 - G(t + x\ h(t))}{1 - G(t)} \\
&= \exp(-x)
\end{aligned}
$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$
\lim_{n \to \infty} \Pr\left\{a_n\left(M_n - b_n\right) \le x\right\} = \exp\left\{-\exp(-x)\right\}
$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

Secondly, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta < 0$ such that

$$
\lim_{t \to \infty} \frac{1 - G(t\,x)}{1 - G(t)} = x^{\beta}
$$

for every $x > 0$. But,

$$
\begin{aligned}
\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} &= \lim_{t \to \infty} \frac{1 - \dfrac{G(tx)}{a + (1-a)G(tx)}}{1 - \dfrac{G(t)}{a + (1-a)G(t)}} \\
&= \lim_{t \to \infty} \frac{1 - \dfrac{G(tx)}{a + 1 - a}}{1 - \dfrac{G(t)}{a + 1 - a}} \\
&= \lim_{t \to \infty} \frac{1 - G(tx)}{1 - G(t)} \\
&= x^{\beta}
\end{aligned}
$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$
\lim_{n \to \infty} \Pr\left\{a_n\left(M_n - b_n\right) \le x\right\} = \exp\left(-x^{\beta}\right)
$$

for some suitable norming constants $a_n > 0$ and b_n.

(2 marks)

Thirdly, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\alpha > 0$ such that

$$\lim_{t \to 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} = x^{\alpha}$$

for every $x > 0$. But,

$$\begin{aligned}
\lim_{t \to 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F(w(F) - t)} &= \lim_{t \to 0} \frac{1 - \dfrac{G\left(w(F) - tx\right)}{a + (1-a)G\left(w(F) - tx\right)}}{1 - \dfrac{G(w(F) - t)}{a + (1-a)G(w(F) - t)}} \\
&= \lim_{t \to 0} \frac{1 - \dfrac{G\left(w(G) - tx\right)}{a + (1-a)G\left(w(G) - tx\right)}}{1 - \dfrac{G(w(G) - t)}{a + (1-a)G(w(G) - t)}} \\
&= \lim_{t \to 0} \frac{1 - \dfrac{G\left(w(G) - tx\right)}{a + 1 - a}}{1 - \dfrac{G(w(G) - t)}{a + 1 - a}} \\
&= \lim_{t \to 0} \frac{1 - G\left(w(G) - tx\right)}{1 - G(w(G) - t)} \\
&= x^{\beta}.
\end{aligned}$$

So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{n \to \infty} \Pr\left\{a_n\left(M_n - b_n\right) \leq x\right\} = \exp\left\{-(-x)^{\alpha}\right\}$$

for some suitable norming constants $a_n > 0$ and b_n.

(2 marks)