MATH3/4/68181: EXTREME VALUES
FIRST SEMESTER
ANSWERS TO IN CLASS TEST

ANSWER TO QUESTION 1 i) If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cdf of a normalized version of M_n converges to G, i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n\left(a_n x + b_n\right) \to G(x) \tag{1}$$

as $n \to \infty$ then G must be of the same type as (cdfs G and G^* are of the same type if $G^*(x) = G(ax+b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$\begin{aligned}
I &: \Lambda(x) = \exp\left\{-\exp(-x)\right\}, \qquad x \in \Re; \\
II &: \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\left\{-x^{-\alpha}\right\} & \text{if } x \geq 0 \end{cases} \\
& \quad \text{for some } \alpha > 0; \\
III &: \Psi_\alpha(x) = \begin{cases} \exp\left\{-(-x)^{\alpha}\right\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \\
& \quad \text{for some } \alpha > 0.
\end{aligned}$$

(3 marks)

ii) The necessary and sufficient conditions for the three extreme value distributions are:

$$\begin{aligned}
I &: \exists \gamma(t) > 0 \text{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F\left(t + x\gamma(t)\right)}{1 - F(t)} = \exp(-x), \qquad x \in \Re, \\
II &: w(F) = \infty \text{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \qquad x > 0, \\
III &: w(F) < \infty \text{ and } \lim_{t \downarrow 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} = x^{\alpha}, \qquad x > 0.
\end{aligned}$$

(3 marks)

iii) First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function, say $h(t)$, such that

$$\lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} = \exp(-x)$$

for every $x \in (-\infty, \infty)$. But, using L'Hopital's rule, we note that

$$\lim_{t \to w(F)} \frac{1 - F\left(t + xh(t)\right)}{1 - F(t)}$$

$$
\begin{aligned}
&= \lim_{t \to w(F)} \frac{\left(1 + xh^{'}(t)\right) f\left(t + xh(t)\right)}{f(t)} \\
&= \lim_{t \to w(G)} \frac{\left(1 + xh^{'}(t)\right) \left\{- \log\left[1 - G\left(t + xh(t)\right)\right]\right\}^2 g\left(t + xh(t)\right)}{\left\{- \log\left[1 - G(t)\right]\right\}^2 g(t)} \\
&= \lim_{t \to w(G)} \frac{\left(1 + xh^{'}(t)\right) g\left(t + xh(t)\right)}{g(t)} \left\{ \frac{\log\left[1 - G\left(t + xh(t)\right)\right]}{\log\left[1 - G(t)\right]} \right\}^2 \\
&= \lim_{t \to w(G)} \frac{\left(1 + xh^{'}(t)\right) g\left(t + xh(t)\right)}{g(t)} \left\{ \frac{1 - G\left(t\right)}{1 - G\left(t + xh(t)\right)} \frac{\left(1 + xh^{'}(t)\right) g\left(t + xh(t)\right)}{g(t)} \right\}^2 \\
&= \lim_{t \to w(G)} \frac{\left(1 + xh^{'}(t)\right) g\left(t + xh(t)\right)}{g(t)} \left\{ \frac{g\left(t\right)}{\left(1 + xh^{'}(t)\right) g\left(t + xh(t)\right)} \frac{\left(1 + xh^{'}(t)\right) g\left(t + xh(t)\right)}{g(t)} \right\}^2 \\
&= \lim_{t \to w(G)} \frac{\left(1 + xh^{'}(t)\right) g\left(t + xh(t)\right)}{g(t)} \\
&= \lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} \\
&= \exp(-x)
\end{aligned}
$$

for every $x \in (-\infty, \infty)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$
\lim_{n \to \infty} \Pr\left\{a_n \left(M_n - b_n\right) \leq x\right\} = \exp\left\{- \exp(-x)\right\}
$$

for some suitable norming constants $a_n > 0$ and b_n.

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$, such that

$$
\lim_{t \to \infty} \frac{1 - G(tx)}{1 - G(t)} = x^{-\beta}
$$

for every $x > 0$. But, using L'Hopital's rule, we note that

$$
\begin{aligned}
\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} &= \lim_{t \to \infty} \frac{x f(tx)}{f(t)} \\
&= \lim_{t \to \infty} \frac{x\left\{- \log\left[1 - G(tx)\right]\right\}^2 g(tx)}{\left\{- \log\left[1 - G(t)\right]\right\}^2 g(t)} \\
&= \lim_{t \to \infty} \frac{x g(tx)}{g(t)} \left\{ \frac{\log\left[1 - G(tx)\right]}{\log\left[1 - G(t)\right]} \right\}^2 \\
&= \lim_{t \to \infty} \frac{x g(tx)}{g(t)} \left\{ \frac{1 - G(t)}{1 - G(tx)} \frac{x g(tx)}{g(t)} \right\}^2 \\
&= \lim_{t \to \infty} \frac{x g(tx)}{g(t)} \left\{ \frac{g(t)}{x g(tx)} \frac{x g(tx)}{g(t)} \right\}^2 \\
&= \lim_{t \to \infty} \frac{x g(tx)}{g(t)}
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{t\to\infty} \frac{1-G(tx)}{1-G(t)} \\
&= x^{\beta}
\end{aligned}
$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$
\lim_{n\to\infty} \Pr\left\{a_n\left(M_n - b_n\right) \le x\right\} = \exp\left(-x^{-\beta}\right)
$$

for some suitable norming constants $a_n > 0$ and b_n.

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\alpha > 0$, such that

$$
\lim_{t\to 0} \frac{1-G(w(G)-tx)}{1-G(w(G)-t)} = x^{\alpha}
$$

for every $x > 0$. But, using L'Hopital's rule, we note that

$$
\begin{aligned}
\lim_{t\to 0} \frac{1-F(w(F)-tx)}{1-F(w(F)-t)} &= \lim_{t\to 0} \frac{xf(w(F)-tx)}{f(w(F)-t)} \\
&= \lim_{t\to 0} \frac{x\left\{-\log\left[1-G(w(F)-tx)\right]\right\}^{a-1} g(w(F)-tx)}{\left\{-\log\left[1-G(w(F)-tx)\right]\right\}^{2} g(w(F)-t)} \\
&= \lim_{t\to 0} \frac{xg(w(F)-tx)}{g(w(F)-t)} \left\{\frac{\log\left[1-G(w(F)-tx)\right]}{\log\left[1-G(w(F)-t)\right]}\right\}^{2} \\
&= \lim_{t\to 0} \frac{xg(w(F)-tx)}{g(w(F)-t)} \left\{\frac{1-G(w(F)-t)}{1-G(w(F)-tx)} \frac{xg(w(F)-tx)}{g(w(F)-t)}\right\}^{2} \\
&= \lim_{t\to 0} \frac{xg(w(F)-tx)}{g(w(F)-t)} \left\{\frac{g(w(F)-t)}{xg(w(F)-tx)} \frac{xg(w(F)-tx)}{g(w(F)-t)}\right\}^{2} \\
&= \lim_{t\to 0} \frac{xg(w(F)-tx)}{g(w(F)-t)} \\
&= \lim_{t\to 0} \frac{1-G(w(F)-tx)}{1-G(w(F)-t)} \\
&= x^{\alpha}.
\end{aligned}
$$

So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$
\lim_{n\to\infty} \Pr\left\{a_n\left(M_n - b_n\right) \le x\right\} = \exp\left\{-(-x)^{\alpha}\right\}
$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

ANSWER TO QUESTION 2 i) Note that $w(F) = \infty$. Then

$$
\lim_{t\uparrow\infty} \frac{1-F\left(t+x\gamma(t)\right)}{1-F(t)} = \lim_{t\uparrow\infty} \frac{1-\left[1-\exp\left(-t-x\gamma(t)\right)\right]^{2}}{1-\left[1-\exp\left(-t\right)\right]^{2}}
$$

$$
\begin{aligned}
&= \lim_{t \uparrow \infty} \frac{2\exp\left(-t - x\gamma(t)\right)}{2\exp\left(-t\right)} \\
&= \exp\left(-x\gamma(t)\right) \\
&= \exp(-x)
\end{aligned}
$$

if $\gamma(t) = 1$. So, the exponentiated exponential cdf $F(x) = [1 - \exp(-x)]^2$ belongs to the Gumbel domain of attraction.

(2 marks)

ii) Note that $w(F) = \infty$. Then

$$
\begin{aligned}
\lim_{t \to \infty} \frac{1 - F\left(t + x\gamma(t)\right)}{1 - F(t)} &= \lim_{t \to \infty} \frac{1 - \dfrac{\left\{1 - \exp\left(-2\left(t + x\gamma(t)\right)\right)\right\}^2}{0.5 + 0.5\left\{1 - \exp\left(-2\left(t + x\gamma(t)\right)\right)\right\}^2}}{1 - \dfrac{\left\{1 - \exp\left(-2t\right)\right\}^2}{0.5 + 0.5\left\{1 - \exp\left(-2t\right)\right\}^2}} \\
&= \lim_{t \to \infty} \frac{1 - \dfrac{\left\{1 - \exp\left(-2\left(t + x\gamma(t)\right)\right)\right\}^2}{0.5 + 0.5}}{1 - \dfrac{\left\{1 - \exp\left(-2t\right)\right\}^2}{0.5 + 0.5}} \\
&= \lim_{t \to \infty} \frac{1 - \left\{1 - \exp\left(-2\left(t + x\gamma(t)\right)\right)\right\}^2}{1 - \left\{1 - \exp\left(-2t\right)\right\}^2} \\
&= \lim_{t \to \infty} \frac{1 - \left\{1 - 2\exp\left(-2\left(t + x\gamma(t)\right)\right)\right\}}{1 - \left\{1 - 2\exp\left(-2t\right)\right\}} \\
&= \lim_{t \to \infty} \frac{2\exp\left(-2\left(t + x\gamma(t)\right)\right)}{2\exp\left(-2t\right)} \\
&= \lim_{t \to \infty} \exp\left(-2x\gamma(t)\right) \\
&= \exp(-x)
\end{aligned}
$$

if $\gamma(t) = 0.5$. So, the exponentiated exponential geometric cdf $F(x) = \frac{\{1-\exp(-2x)\}^2}{1-0.5+0.5\{1-\exp(-2x)\}^2}$ belongs to the Gumbel domain of attraction.

(2 marks)

iii) For the geometric distribution,

$$
F(k) = 1 - (1 - p)^k,
$$

so

$$
\frac{\Pr(X = k)}{1 - F(k - 1)} = \frac{p(1 - p)^{k-1}}{(1 - p)^{k-1}} = p.
$$

Hence, there can be no sequences $a_n > 0$ and b_n such that $(M_n - b_n)/a_n$ has a non-degenerate limiting distribution.

(2 marks)

iv) Note that $w(F) = \infty$. Then

$$
\begin{aligned}
\lim_{t\to\infty} \frac{1 - F(t + xg(t))}{1 - F(t)} &= \lim_{t\to\infty} \frac{1 - \Phi^2(t + xg(t))}{1 - \Phi^2(t)} \\
&= \lim_{t\to\infty} \frac{-2\Phi(t + xg(t))\,\phi(t + xg(t))\left(1 + xg'(t)\right)}{-2\Phi(t)\phi(t)} \\
&= \lim_{t\to\infty} \frac{\Phi(t + xg(t))\,\phi(t + xg(t))\left(1 + xg'(t)\right)}{\Phi(t)\phi(t)} \\
&= \lim_{t\to\infty} \frac{\phi(t + xg(t))\left(1 + xg'(t)\right)}{\phi(t)} \\
&= \lim_{t\to\infty} \frac{\frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(t + xg(t))^2}{2}\right]\left(1 + xg'(t)\right)}{\frac{1}{\sqrt{2\pi}} \exp\left[-\frac{t^2}{2}\right]} \\
&= \lim_{t\to\infty} \frac{\exp\left[-\frac{(t + xg(t))^2}{2}\right]\left(1 + xg'(t)\right)}{\exp\left[-\frac{t^2}{2}\right]} \\
&= \lim_{t\to\infty} \exp\left[\frac{t^2}{2} - \frac{(t + xg(t))^2}{2}\right]\left(1 + xg'(t)\right) \\
&= \lim_{t\to\infty} \exp\left\{-\frac{1}{2}\left[(t + xg(t))^2 - t^2\right]\right\}\left(1 + xg'(t)\right) \\
&= \lim_{t\to\infty} \exp\left\{-\frac{1}{2}\left[2txg(t) + x^2g^2(t)\right]\right\}\left(1 + xg'(t)\right) \\
&= \lim_{t\to\infty} \exp\left\{-x - \frac{x^2}{2t^2}\right\}\left(1 - \frac{x}{t^2}\right) \\
&= \exp(-x)
\end{aligned}
$$

if $g(t) = 1/t$. So, F belongs to the Gumbel domain of attraction.

(2 marks)

v) Note that $w(F) = \infty$. Then

$$
\begin{aligned}
\lim_{t\uparrow\infty} \frac{1 - F(t + xg(t))}{1 - F(t)} &= \lim_{t\uparrow\infty} \frac{1 - \exp\{-\exp(-t - xg(t))\}}{1 - \exp\{-\exp(-t)\}} \\
&= \lim_{t\uparrow\infty} \frac{1 - 1 + \exp(-t - xg(t))}{1 - 1 + \exp(-t)} \\
&= \lim_{t\uparrow\infty} \frac{\exp(-t - xg(t))}{\exp(-t)} \\
&= \lim_{t\uparrow\infty} \exp(-t - xg(t))
\end{aligned}
$$

$$= \exp(-x)$$

if $g(t) = 1$. So, the cdf $F(x) = \exp\{-\exp(-x)\}$ belongs to the Gumbel domain of attraction.

(2 marks)

ANSWER TO QUESTION 3 i) The cdf of Y is

$$\begin{aligned} F_Y(y) &= \Pr(Y \leq y) \\ &= \Pr\left(\max\left(X_1, \ldots, X_\alpha\right) \leq y\right) \\ &= \Pr\left(X_1 \leq y, \ldots, X_\alpha \leq y\right) \\ &= \Pr\left(X_1 \leq y\right) \cdots \Pr\left(X_\alpha \leq y\right) \\ &= \left[1 - \left(\frac{K}{y}\right)^a\right] \cdots \left[1 - \left(\frac{K}{y}\right)^a\right] \\ &= \left[1 - \left(\frac{K}{y}\right)^a\right]^\alpha . \end{aligned}$$

(2 marks)

ii) The corresponding pdf is

$$f_Y(y) = a\alpha K^a y^{-a-1} \left[1 - \left(\frac{K}{y}\right)^a\right]^{\alpha-1}$$

for $y \geq K$.

(2 marks)

iii) The nth moment of Y can be calculated as

$$\begin{aligned} E\left(Y^n\right) &= a\alpha K^a \int_K^\infty y^{n-a-1} \left[1 - \left(\frac{K}{y}\right)^a\right]^{\alpha-1} dy \\ &= a\alpha K^a \int_K^\infty K^{n-a-1} (1-x)^{-\frac{n-a-1}{a}} x^{\alpha-1} \frac{K}{a} (1-x)^{-\frac{1}{a}-1} dx \\ &= \alpha K^n \int_0^1 x^{\alpha-1} (1-x)^{-n/a} dx \\ &= \alpha K^n B\left(\alpha, 1 - \frac{n}{a}\right) \end{aligned}$$

provided that $n < a$, where we have set $x = 1 - (K/y)^a$. So,

$$E\left(Y\right) = \alpha K B\left(\alpha, 1 - \frac{1}{a}\right).$$

(3 marks)

iv) So,

$$Var\left(Y\right) = \alpha K^2 B\left(\alpha, 1 - \frac{2}{a}\right) - \alpha^2 K^2 B^2\left(\alpha, 1 - \frac{1}{a}\right).$$

(3 marks)